Optimal Portfolio Allocation Under a Probabilistic Risk Constraint and the Incentives for Financial Innovation*

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June 2001

Keywords: Portfolio Optimization, Value–at–Risk, NP-hard

Abstract

We derive, in a complete markets environment, an investor's optimal portfolio allocation subject to both a budget constraint and a probabilistic risk constraint. We demonstrate that the set of feasible portfolios need not be connected or convex, while the number of local optima increases exponentially with the number of securities implying that finding the optimal portfolio is computationally complex (NP hard). The resulting optimal portfolio allocation may not be monotonic in the state–price density. A novel type of financial innovation, which splits states of nature, is shown to weakly enhance welfare, restore monotonicity in the state–price density, and may reduce complexity.

*We are grateful to Ken Kavajecz and seminar participants at Harvard Business School, London School of Economics, Maastrict University, ZEI Bonn, and Danske Bank Symposium on Asset Allocation and Value-at-Risk: Where Theory Meets Practice for comments on an earlier version of this paper. Corresponding author: Bjorn N. Jorgensen, Morgan Hall 413, Soldiers Field, Harvard Business School, Boston, MA 02163. Ph: +1 (617) 495-5976. E-mail: *bjorgensen@hbs.edu*. Danielsson's mail is *j.danielsson@lse.ac.uk*, de Vries's e-mail is *cdevries@few.eur.nl*, and Yang's e-mail is *xiaoguangy@yahoo.com*.

1 Introduction

Economists have grappled formally with the problem of portfolio optimization subject to a budget constraint since Markowitz (1952). It is however of interest to embed this problem in a richer context, such as the introduction of an additional constraint on possible losses. Examples are the safety–first criterion (Roy, 1952) and the closely related concept of portfolio insurance, (Leland, 1980) which rules out sufficiently adverse outcomes with probability one. More recently, a different type of constraint has become popular within the banking sector and regulators: Value–at–Risk (VaR)[1] where the probability of adverse outcomes is restricted. We demonstrate that under this probabilistic risk constraint the portfolio problem is NP–hard[2], and hence computationally intractable. We proceed to show that investors can simplify their portfolio allocation problem and increase welfare through a financial innovation that entails introducing additional *redundant* securities which are lotteries over existing elementary Arrow–Debreu securities. By splitting existing states, we show that not only is utility increased, but also that an otherwise NP–hard problem can easily be rendered solvable.

The particular setting we consider is a single period complete market spanned by n Arrow-Debreu securities. The investor with initial wealth, W, can purchase, for price p_i at time, zero contingent claims on each state, $i = 1, .., n$, which pay out \$1 in state i at the investment horizon and zero otherwise. State i occurs with probability π_i. We consider an investor whose preferences are characterized by a strictly increasing, concave expected utility function in wealth. Then the investor's problem is choosing portfolio $\{x_i\}_{i=1}^n$:

$$\begin{array}{ll} \max\limits_{\{x_i\}_{i=1}^n} & \sum_{i=1}^n \pi_i u\left(x_i\right) \\ \text{s.t.} & \sum_{i=1}^n p_i x_i \leq W \\ & \sum_{i=1}^n 1_{\{x_i \leq K\}} \pi_i \leq \delta \end{array}$$

where K is the VaR level and δ the risk level. This implies that the probability that the value of the investor's portfolio falls below K at the investment horizon cannot exceed δ. While we consider any value of $\delta \in [0, 1]$, most preceding literature has focused on the two extreme values, $\delta = 1$ and $\delta = 0$.

Consider first the case where $\delta = 1$, such that the risk constraint is never

[1] The VaR problem has received considerable interest, not the least since it forms the foundation of market risk regulations (see Basel Committee on Banking Supervision, 1996).

[2] A problem is Non–deterministic Polynomial (NP)–complete when no other NP problem is more than a polynomial factor harder and any problem in the class of NP-complete is NP–hard. For example, the Travelling Salesman problem is NP–hard.

binding. This is the traditional portfolio allocation setting. Solving such problems does not introduce any significant difficulty as standard methodology readily applies, see e.g. (Kuhn and Tucker, 1951; Uzawa, 1958), and optimum is ensured for strictly concave utility functions. It follows that this portfolio problem is readily solved in polynomial time.

The second case is portfolio insurance where $\delta = 0$ and corresponds to a deterministic floor constraint where losses cannot fall below the exogenously determined level, K. Grossman and Vila (1989) demonstrate that the solution to this problem is straightforward, where in effect the investor buys a put option with exercise price K. We demonstrate below that the portfolio insurance problem falls into the P class and hence is solvable in polynomial time. An early example of a non–deterministic constraint is the safety–first portfolio selection program as analyzed in the Arzac and Bawa (1977) lexicographic interpretation. However, safety–first may lead to strained portfolio choices: A portfolio manager may allocate just enough to the safe asset to meet the downside risk constraint, while the remainder of funds is invested in the option with the highest strike price available (see Dert and Oldenkamp, 1997; Vorst, 2000).

The general case considered in this paper, allows δ to take any value between zero and one. This is the VaR problem where the investor does not want wealth to fall below the level K with more than a given probability δ. We demonstrate in Theorem 1 that this problem belongs to the class of NP–hard problems, implying the problem is computationally complex unless the number of states of nature is sufficiently small or the VaR constraint is non–binding. The reason for the complexity is that the set of feasible portfolios is no longer convex, in fact, the budget set has disjoint components, each containing a local optimum. In an economy with few states it is straightforward to consider all components to find the solution. However, in general, the number of local maxima can increase exponentially in the number of states, i.e., the number of local maxima becomes potentially $n!$.

We demonstrate below how such a non–deterministic downside risk constraint may also change the qualitative aspect of the solution in an important way. It is well known that the optimal portfolio resulting when $\delta = 0$ or $\delta = 1$ is monotonic in the state price density. In a recent paper, Basak and Shapiro (2001) demonstrate that the VaR constrained optimal portfolio is monotonic in the state price density in a Brownian motion setting with continuous rebalancing. In contrast, we show this feature of the solution is lost with the introduction of the VaR constraint when the state space is discrete. Nevertheless, for the discrete state space problem we derive a sufficient condition, i.e. a restriction on the state price density, such that the optimal portfolio of the discrete problem is monotonic in the state price density. Moreover, we show that investors have an incentive by means of financial innovation to create apparently redundant securities to meet this condition. These new securities also render the initial NP-hard problem easily solvable, since the monotonicity in the state price density guarantees polynomial time solutions.

The literature on financial innovations recognizes three distinct categories of innovation. Initial models document how financial innovation is used to complete markets, see Ross (1976). The second category strictly stays within the incomplete markets paradigm, i.e., even after the introduction of new securities the market remains incomplete. In this case, some investors have an incentive to issue new securities, but this need not improve efficiency, (see e.g. Allen and Gale, 1988, 1991). On a general level, these two categories of financial innovation thus reflect the issues discussed in the second best literature. A third category discusses financial innovation in a complete market setting where the marketable securities, which are bundles of (non-traded) elementary securities, achieve complete spanning. In this case regulation prohibits unlimited trading in available securities. Miller (1986, 1992) argues that financial innovation is often driven by regulation, e.g. short sale restrictions. Such trading friction gives an incentive for financial innovation via unbundling of states, in order to achieve the Pareto optimal free market outcome through trading the elementary securities. A representative example of this approach is Chen (1995). Recently Grinblatt and Longstaff (2000) in an empirical study argue that stripping and trading separately the principal and coupons of treasury bond helps in completing markets and overcoming frictions (which explains why rebundling is observed simultaneously with unbundling).

In this paper we introduce a fourth category of financial innovations in complete markets, where all elementary securities are freely traded. We demonstrate that there may nevertheless remain an incentive for financial innovation whereby elementary securities are split up and new states are artificially created through a randomizing devise. We show that the demand for this type of innovation is triggered by the imposition of non–deterministic downside risk constraints, such as VaR. In this case the splitting of states can help in reducing the negative effects of risk constraints on individual expected utility. An example of this type of securities, discussed below, is the lottery bonds issued by many European governments. Lottery bonds are bonds for which part of the issue is called early through a randomizing device. We show that such securities may simplify the investor's portfolio allocation problem.

The condition under which the VaR constrained problem becomes simple is the special case where the probabilities and payoffs can be weakly reversely ordered, i.e., $\pi_i \leq \pi_j$ iff $p_i \geq p_j$ for all $i, j = 1, 2, ..., n$. This means that the states can be ordered such that $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_n$ while $p_1 \geq p_2 \geq \cdots \geq p_n$. We demonstrate that under this *reverse ordering condition* we can write the risk constraint as a restriction on price p_i only.[3] In effect under this condition, the VaR constrained problem is reduced to a generalized portfolio insurance problem. Consider the special case of uniform probabilities. Since probabilities do not play a role for any δ, the intuition is immediate. Under the reverse ordering condition, we show that the risk constrained problem enables polynomial time solutions by employing techniques similar as Grossman and Vila (1989). Without the reverse ordering condition, utility maximization subject to the probabilistic VaR constraint falls into the category of NP–hard problems. If however, state splitting is possible, e.g., there are no regulatory hurdles, we show it is possible to split the states such that the reverse ordering condition is satisfied. Relative to the augmented state space, the problem has a polynomial time solution. We also demonstrate the positive impact of state splitting in an example where the VaR constraint is met optimally by investing in one of the two new states. Subsequent to state splitting, the optimal portfolio is monotonic in the state price density.

The structure of the paper is as follows. In Section 2 we demonstrate that the risk constrained problem is NP–hard. Following in Section 3 we introduce the reverse ordering condition and discuss portfolio insurance and uniformly distributed states. In Section 4 we allow for financial innovation and document their benefits. Section 5 concludes the paper.

2 Utility Maximization and Risk Constraints

2.1 The Standard Portfolio Problem

Consider first the standard portfolio choice problem (1) with $\delta = 1$, which we refer to as the unconstrained case because the VaR constraint is never binding. Let λ denote the Lagrange multiplier associated with the budget constraint in (1) which is binding. An interior solution to (1), if it exists, is characterized by the budget constraint and

$$x_i = (u')^{-1} (\lambda p_i / \pi_i), \forall i. \tag{2}$$

[3] Instead of also involving probabilities π_i.

Without loss of generality, the states can then be ordered such that

$$\lambda p_i/\pi_i \geq \lambda p_{i+1}/\pi_{i+1}\,,\ \forall i \tag{3}$$

or, equivalently, such that

$$x_i \leq x_{i+1}\,,\ \forall i.$$

Given the assumptions above, the first–order conditions guarantee that (2) is a globally optimal solution when the utility function is strictly concave and the domain is a bounded convex set.

Consider second the portfolio choice problem (1) with $\delta = 0$. Grossman and Vila (1989) demonstrate that this portfolio insurance problem has the intuitively appealing feature that the investor buys a put with exercise price K. The more recent Value–at–Risk (VaR) constraint, discussed in the next Section, is an extended portfolio insurance problem where the investor's net worth cannot fall below K with more than δ probability.

2.2 Risk Restrictions

Public policy often restricts the risk that can be assumed by a strict subset of all investors. For example, national supervisory authorities impose VaR constraints on commercial banks in order to contain market risk. Other financial intermediaries like pension funds work with self–imposed constraints like portfolio insurance where the investors' net worth is never allowed to fall below a certain predetermined level K.

The formal definition of the VaR constraint is as follows.

Definition 1 (Value–at–Risk). VaR is the zero'th lower partial moment. For any discrete distribution, the VaR constraint can be written as

$$\textstyle\sum_{i=1}^{n} 1_{\{x_i \leq K\}}\pi_i \quad \leq \quad \delta$$

where δ is the associated probability level.

To ensure the existence of a finite solution to the investor's portfolio choice problem, $|x_i| < \infty, \forall i$, two different approaches have been suggested. First, one can rule out unlimited borrowing. For presentational simplicity, we simply rule out all short sales in Arrow-Debreu securities. However, all results go through with an arbitrary, finite lower bound on short selling. Moreover note that this does *not* constrain investors from short selling non-elementary securities. Also note that this does not affect the feasibility of the VaR problem.[4] Then the investor's problem can be stated as:

Problem 1.

$$\max_{\{x_i\}_{i=1}^n} \sum_{i=1}^n \pi_i u(x_i) \qquad \text{s.t.} \qquad \begin{array}{rcl} \sum_{i=1}^n p_i x_i & \leq & W \\ \sum_{i=1}^n 1_{\{x_i \leq K\}} \pi_i & \leq & \delta \\ x_1, x_2, \cdots, x_n & \geq & 0 \end{array}$$

Alternatively, we could impose mild regularity conditions on the investor's utility function.[5] Since all examples below adhere to these regularity conditions, the short selling restriction is not driving our results.

2.3 Solution Complexity and NP–hardness

In order to characterize the computational complexity, we relate the current Problem 1 to the partition problem since the partition problem is known to be NP–complete, and finding the solution to the partition problem is NP–hard.

Definition 2 (Partition Problem). Let $\{a_1, a_2, \cdots, a_n; b\}$ be $n+1$ positive numbers such that $\sum_{i=1}^n a_i = 2b$. Does a subset $S \subset \{1, 2, \cdots, n\}$ exist such that $\sum_{i \in S} a_i = b$?

Theorem 1. *The feasibility of Problem 1 is NP-complete.*

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[4] Ruling out infinite shortsales is equivalent to preventing infinite borrowing. Impose restriction, B, on borrowing then the investor's problem can be stated as:

$$\min_{\{x_i\}_{i=1}^n} \sum_{i=1}^n 1_{\{x_i \leq K\}} \pi_i \qquad \text{s. t.} \qquad \begin{array}{rl} \sum_{i=1}^n p_i x_i & = W \\ x_1, x_2, \cdots, x_n & \geq -B \end{array}$$

Let $y_i = x_i + B$, then it is equivalent to

$$\min_{\{y_i\}_{i=1}^n} \sum_{i=1}^n 1_{\{y_i \leq B+K\}} \pi_i \qquad \text{s. t.} \qquad \begin{array}{rl} \sum_{i=1}^n p_i y_i & = W + B \sum_{i=1}^n p_i \\ y_1, y_2, \cdots, y_n & \geq 0 \end{array}$$

which for the purpose of the arguments in this paper is equivalent to ruling out short sales.

[5] Such as either the Inada conditions or the self–concordance condition, see Ingersoll (1987, p. 189), and den Hertog (1995) respectively.

Proof. It is trivial to see that the feasibility of Problem 1 is NP. To establish NP–completeness, we only need show that the problem can be reduced to the partition problem. From a given instance of the partition problem, let us construct an instance of Problem 1. Let $\pi_i = \frac{a_i}{2b}, p_i = a_i,\ 1 \le i \le n$. Let $K = b, W = b^2$ and $\delta = \frac{1}{2}$. We claim that the partition problem has a feasible solution if and only if Problem 1 has a feasible solution. In fact, if the partition problem has a feasible solution S, that is $\sum\limits_{i \in S} a_i = b$, we set $x_i = b$ if $i \in S$ and $x_i = 0$ otherwise. Then we have $\sum\limits_{i=1}^{n} p_i x_i = b^2$, $\sum\limits_{x_i \ge K} \pi_i = \frac{1}{2}$. Thus x is a feasible solution of Problem 1. Conversely, assume x is a feasible solution of Problem 1. We have $\sum\limits_{i=1}^{n} p_i x_i = b^2$ and $\sum\limits_{x_i \ge K} \pi_i \ge \frac{1}{2}$. Denote $S = \{1 \le i \le n | x_i \ge K\}$. We have

$$b^2 = \sum_{i=1}^{n} p_i x_i \ge \sum_{i \in S} p_i b$$

$$\sum_{i \in S} \pi_i = \sum_{x_i \ge K} \pi_i \ge \frac{1}{2}$$

Using $p_i = a_i$ and $\pi_i = a_i/2b$ we get

$$b \ge \sum_{i \in S} a_i,$$

$$\sum_{i \in S} a_i \ge b.$$

Hence we obtain $\sum\limits_{i \in S} a_i = b$. This means that S is a feasible solution of the partition problem. □

It follows that finding the solution to the VaR constrained portfolio choice Problem 1 is NP–hard. Hence, the optimal portfolio can be computed when n is sufficiently small or the problem is otherwise simple (see below). In general the number of local maxima becomes potentially $n!$, i.e., the number of potential local maxima increases exponentially in the number of states.

Note that the proof of Theorem 1 allows for a risk neutral investor with limited short sales. Thus, when the lexicographic interpretation of the safety-first criterion is feasible, it is also burdened by the computational complexity.

2.4 Characterizing the Optimal Portfolio

Example 1. Consider an economy with two states of nature, a VaR constraint and logarithmic utility.

$$\begin{array}{lrcl} \max_{x_1,x_2} \pi_1 \log(x_1) + \pi_2 \log(x_2) & & & \\ \text{s.t.} & p_1 x_1 + p_2 x_2 & \leq & W \\ & 1_{\{x_1 \leq K\}} \pi_1 + 1_{\{x_2 \leq K\}} \pi_2 & \leq & \delta \end{array}$$

Among the three possible ordering of probabilities and constraints, we consider the case where $1 > \delta > \max(\pi_1, \pi_2) > 0$, see Figure 1. In the Figure L represents the unconstrained portfolio allocation that would arise in the absence of a VaR constraint, which of course it violates. Due to the VaR constraint, the relevant budget constraint is the two disjoint line segments $(0, W/p_1)$—N and M—$(W/p_2, 0)$. These segments are individually convex, but not jointly, hence the Kuhn–Tucker conditions may only give a local solution, since there are two local optima.

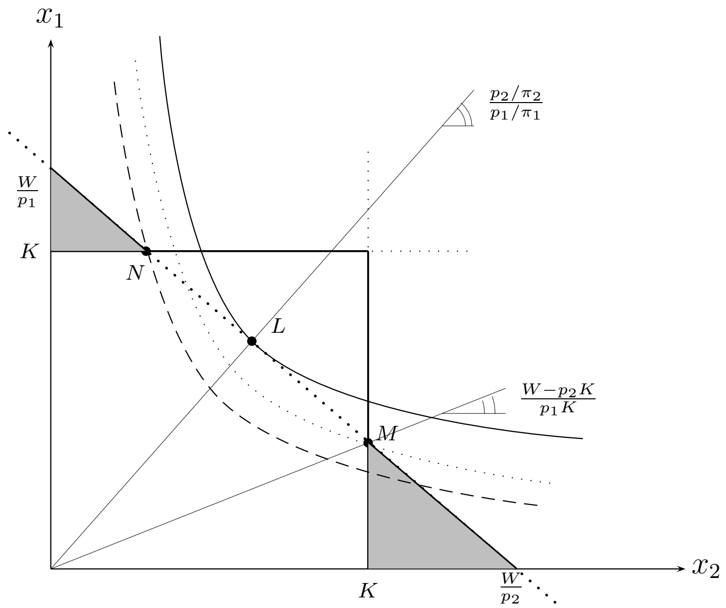

Figure 1: Concave utility and the VaR constraint
Either x_1 or x_2 must satisfy the risk level K. The budget constraint is between points W/p_1 and W/p_2. Due to the shortsale restriction, negative values are ruled out, and the line segment between N and M is not feasible due to the VaR constraint. Hence the budget constraint is the two disjoint pieces W/p_1—N and M—W/p_2.

We can write the VaR constraint as a product, since either $x_1 \geq K$ or $x_2 \geq K$:

$$\begin{aligned}\mathcal{L} &= \pi_1 log(x_1) + \pi_2 log(x_2) \\ &\quad +\lambda\{W - p_1 x_1 - p_2 x_2\} \\ &\quad +\psi\{-(x_1 - K)(x_2 - K)\}.\end{aligned}$$

The necessary first order conditions are

$$\begin{aligned}\pi_1 \frac{1}{x_1} - \lambda p_1 - \psi(x_2 - K) &= 0 \\ \pi_2 \frac{1}{x_2} - \lambda p_2 - \psi(x_1 - K) &= 0 \\ \lambda\{W - p_1 x_1 - p_2 x_2\} &= 0 \\ \psi\{-(x_1 - K)(x_2 - K)\} &= 0 \\ \lambda \geq 0, \psi &\geq 0.\end{aligned}$$

Consider a specific parameterization. Assume $\pi_1 = 2/3$, $2/3 < \delta < 1$, $p_1 = 1/3$, $p_2 = 1/4$, $W = 1/4$, and $K = 23/32$. In the unconstrained case, the optimal portfolio allocation would be $x_1 = 1/2$ and $x_2 = 1/3$, which is represented by the point L in Figure 1. Note that this portfolio does not fulfill the VaR constraint. To achieve this, suppose we adjust the cheaper state, i.e. raise x_1 to K (this appears to be cheaper because x_1 is closer to K than x_2 and has a lower state price probability ratio, since $p_1/\pi_1 = 1/2 < p_2/\pi_2 = 3/4$). This is labelled as N in Figure 1. Alternatively, we could raise x_1 to K, the resulting portfolio allocation is labelled as M in Figure 1. The following Table summarizes the portfolio compositions and expected utilities:

Maxima	x_1	x_2	EU
Unconstrained (L)	1/2	1/3	−0.83
VaR constrained (N)	23/32	1/24	−1.28
VaR constrained (M)	27/128	23/32	−1.15

The unconstrained solution L favors x_1 over x_2 since $p_1/\pi_1 < p_2/\pi_2$ (the usual monotonicity property). The optimal VaR constrained portfolio upsets the usual ordering. Note that in the optimal VaR constrained solution, the ratio x_1/x_2 is only a function of prices, p_1 and p_2, and the VaR constraint K. In contrast, both prices and probabilities determine this ratio in the unconstrained solution. Since probabilities do not determine the x_1/x_2 ratio in the VaR constrained case, the monotonicity suggested by the state price density can be upset. For this reason all constrained local optima have to be evaluated for finding the global maximum. For a more general problem, it is possible to write a unified Lagrangian function with probabilistic integer variables. However, since such integer variables are non differentiable, solutions based on first order conditions are not feasible. With many states this remains a cumbersome exercise.

With just two states we cannot fully do justice to the qualitative properties of the VaR constrained solution. Therefore, we also give a four–state example.

Example 2. Let the investor's utility function be $u(x) = -1/x$, which implies a relative risk aversion parameter of 2. Let there be four states. The following table lists the state probabilities and state prices of the four elementary securities.

Statistic	States			
i	1	2	3	4
π_i	1/100	1/2	9/100	40/100
p_i	1/100	1/8	1/100	1/40

Suppose wealth $W = 39/90$. It is then a simple matter to verify that, in the absence of a VaR constraint, the investor chooses $(x_1, x_2, x_3, x_4) = (10/9, 20/9, 30/9, 40/9)$, which yields expected utility $EU = \sum_{i=1}^{4} \pi_i u(x_i) = -0.3510$. Note that since the state price density is $(p_1/\pi_1, p_2/\pi_2, p_3/\pi_3, p_4/\pi_4) = (1, 1/4, 1/9, 1/16)$, the solution $x_1 \leq x_2 \leq x_3 \leq x_4$ is in conformity with (3). Suppose the VaR regulation stipulates $\delta = 0.4999$ and $K = 2.5$. The unconstrained solution does not meet the downside risk constraint, since both x_1 and x_2 are below 2.5. Following the rule of thumb implied by the state price density ordering, one would raise the consumption in the state 2, as x_2 is closest to the VaR level 2.5. In that case the investor chooses $(x_1, x_2, x_3, x_4) \approx (0.86, 2.50, 2.59, 3.45)$, with expected utility $EU = \sum_{i=1}^{4} \pi_i u(x_i) \approx -0.3622$. Note this portfolio allocation is feasible and preserves the ordering of the state price density. This portfolio is, however, not the optimal solution. The investor is better off by taking $x_1 = 2.5$. This permits $(x_1, x_2, x_3, x_4) \approx (2.50, 2.14, 3.21, 4.28)$, with expected utility $EU = \sum_{i=1}^{4} \pi_i u(x_i) \approx -0.3576$. Although this is the optimal VaR restricted portfolio, the allocation is not monotonic with respect to the state price density since $x_1 > x_2 < x_3 < x_4$.

Statistic	States				EU
	1	2	3	4	
π_i	1/100	1/2	9/100	40/100	
p_i	1/100	1/8	1/100	1/40	
x_i unconstrained	1.11	2.22	3.33	4.44	−0.3510
x_2 constrained	0.86	2.5	2.59	3.45	−0.3622
x_1 constrained	2.5	2.14	3.21	4.28	−0.3576

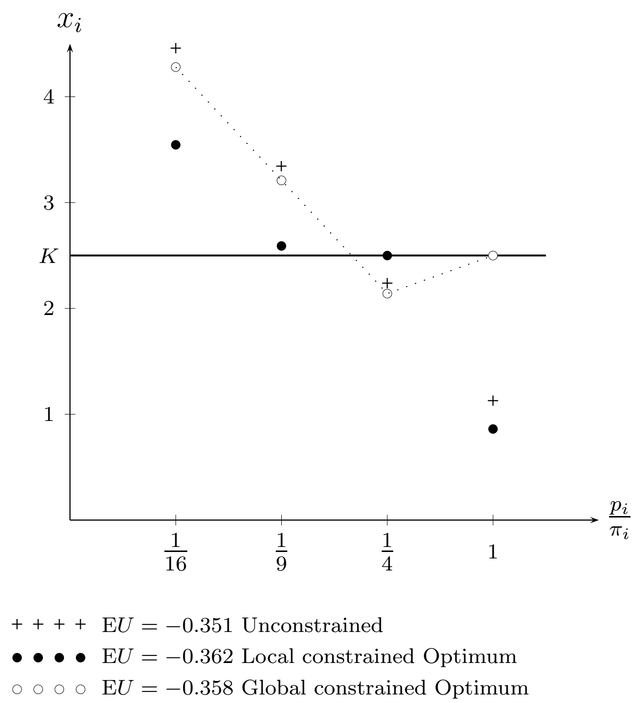

Figure 2: Monotonicity of solution

In the examples above the monotonicity of the optimal unconstrained allocation with respect to the state price density is upset in the optimal VaR constrained allocation. In Figure 2 the dotted line shows that the optimal VaR constrained portfolio is in fact non–monotonic with respect to the state price density. That is, the investor would be strictly worse off by restricting attention to portfolios that exhibit monotonicity with respect to the state price density. Interestingly, Basak and Shapiro (2001) have recently shown in a continuous time setting, that the VaR constrained portfolio allocation is still monotonic with respect to the continuous state price density derived from underlying Brownian motion. Example 2 shows that the discrete case can be qualitatively different.

3 Conditions for Computational Simplicity

Given that the investor's portfolio problem is NP–hard, it is valuable to identify special cases which are easily solvable, i.e. P.

3.1 Portfolio Insurance

Portfolio insurance is a special case of Problem 1 where $\delta = 0$ that can be solved in polynomial time.

Corollary 1. *Portfolio insurance is in the class P.*

Proof. Grossman and Vila (1989) demonstrate that the problem is solved by implementing a put at the desired VaR–level while the consumption in each state is reduced to finance this put, in keeping with the first order conditions of the unconstrained problem at the reduced wealth level. Order the state consumption of the unconstrained problem in descending order $x_1 \geq \ldots \geq x_n$. Suppose the VaR restriction becomes binding at $x_m < x_1$. A sequential search procedure involving at the most $n - m$ steps yields the optimal policy. First check whether the put that ensures consumption at the VaR level in states $m, \ldots, n$ can be financed by reducing the consumption in the remaining uninsured states. If this implies that consumption in state x_{m-1} drops below the VaR level, restart the procedure by insuring consumption in states $m - 1, m, \ldots, n$ at the VaR level. Repeat this until the VaR level is met or exceeded in all states. □

3.2 Uniform Distribution

Suppose the probability distribution of the states is discrete uniform, i.e., $\pi_1 = \pi_i, \forall i$. In that case, the probabilities clearly do not enter in determining the optimal portfolio allocation. Only prices of the states matter, with or without the downside risk constraint in place. Recall Example 1 and Figure 1, which showed that prices and the VaR level K determine the optimum allocation in the downside risk constrained equilibrium, while prices and probabilities determine the portfolio choice in the unconstrained equilibrium. This latter conclusion is now replaced by recognizing that only prices determine the allocation in both problems. In the unconstrained case if prices are ordered by $p_1 \leq p_2 \ldots \leq p_n$, then $x_n \geq x_2 \geq \ldots \geq x_n$. For the VaR constrained case, the program for finding the optimal solution preserves this monotonicity, and hence trivially the monotonicity with respect to the state price density, and is, moreover, easily implemented. To find the optimal VaR constrained allocation, first compute the unconstrained allocation. Subsequently, start tracing the allocation up to K for the state with the lowest price which falls below K in the unconstrained solution. Then turn to the next cheapest state and raise it to K. Repeat this procedure until the sum of the probabilities of the (most expensive) states which have $x_i < K$ is less than or equal to the desired level δ. This takes at most $n-1$ steps.

Example 3. Continuing with Example 1, assume state 1 has been split into two equally probable states, labelled $1A$ and $1B$, each with half the price of the former state 1 security. We now have a three state economy with $\pi_{1A} = \pi_{1B} = \pi_2 = 1/3$. Assume, as before, $U(x) = log(x)$, let $W = 1/4$, $K = 23/32$, $p_2 = 1/4$ but $p_{1A} = p_{1B} = 1/6$. It is easily verified that the expected utility, EU, of the unconstrained case and the VaR constrained case with $x_2 = 23/32$ is identical to the EU in Example 1, since the prices and probabilities of states 1 and 2 in the new example are exactly half the price and probability of state 1 in Example 1.

	x_{1A}	x_{1B}	x_2	EU
unconstrained	1/2	1/2	1/3	−0.83
x_2 constrained	27/128	27/128	23/32	−1.15
x_{1A} constrained	23/32	23/64	23/96	−0.93

However, since state 1 is now broken into two cheaper states, while state 2 has the price and probability parameters of Example 1, it is now optimal to place the VaR constraint on state 1A (or, alternatively state 1B), rather than raising the consumption of the more expensive state 2. To meet the VaR constraint, the investor now minimizes the expenditure by raising the allocation in those states which are closest to the VaR level as these are least costly.

3.3 Reverse Order Condition

We now generalize the previous case of uniform probabilities. Suppose that prices and probabilities are weakly inversely ordered. Note that the previous case of uniform distribution fulfills this inverse order condition. As we argue below, this case is highly interesting from an economic point of view, since it may be the result of financial innovation. Formally the condition is:

Condition 1. *Problem 1 satisfies the reverse order condition if there exists a labelling of the states such that the pairs* (π_i, p_i) *can be ordered such that*

$$\begin{aligned} \pi_1 &\leq \pi_2 \leq \cdots \leq \pi_n \\ p_1 &\geq p_2 \geq \cdots \geq p_n \end{aligned}$$

Remark 1. Note that the reverse ordering condition labelling satisfies the complete market equilibrium condition (3).

Theorem 2. *For any* δ*, let* q *be the number such that*

$$\sum_{i=1}^{q} \pi_i \leq \delta < \sum_{i=1}^{q+1} \pi_i. \tag{4}$$

If (π, p) *satisfies Condition 1 then Problem 1 can be transformed as follows:*

$$\max \sum_{i=1}^{n} \pi_i u(x_i) \qquad \text{s.t.} \qquad \begin{aligned} \sum_{i=1}^{n} p_i x_i &= W \\ 0 \leq x_1 \leq x_2 \leq \cdots &\leq x_n \\ x_{q+1} &\geq K \end{aligned} \tag{5}$$

Proof. See Appendix A. □

Theorem 3. *The solution program to (5) is in P.*

Proof. Note that (5) is a linear constrained optimization problem. The feasibility problem of (5) is easy. We need only judge whether the inequality system has solutions. To this end, consider the following linear program,

$$\max x_{q+1} \quad \text{s.t} \qquad \begin{aligned} \sum_{i=1}^{n} p_i x_i = W \\ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \end{aligned} \tag{6}$$

It is straightforward to see that the optimal value of (6) is $W/(p_{q+1}+\cdots+p_n)$. Therefore Problem 1 is feasible under Condition 1 if and only if $W/(p_{q+1}+\cdots+p_n) \geq K$. Consider the following constrained problem:

$$\max \sum_{i=1}^{n} \pi_i u(x_i) \quad \text{s.t} \quad \begin{array}{rcl} \sum_{i=1}^{n} p_i x_i & = & W \\ x_i & \geq & 0, \quad i = 1, 2, \cdots, q \\ x_i & \geq & K, \quad i = q+1, q+2, \cdots, n \end{array} \tag{7}$$

If $u(x)$ is strictly concave, following Grossman and Vila (1989), we can prove that there exists $W' \leq W$, such that

$$h_i(W') = \begin{cases} x_i(W') & i = 1, 2, \cdots, q \\ x_i(W') + \max\{K - x_i(W'), 0\} & i = q+1, q+2, \cdots, n \end{cases} \tag{8}$$

and $\{h_i(W')\}$ is an optimal solution of (7). In (8) $\{x_i(W')\}$ is the optimal solution of the following simple portfolio optimization problem

$$\max \sum_{i=1}^{n} \pi_i u(x_i) \quad \text{s.t} \quad \begin{array}{rcl} \sum_{i=1}^{n} p_i x_i & = & W' \\ x_1, x_2, \cdots, x_n & \geq & 0 \end{array}$$

Under Condition 1, we have $0 \leq h_1(W') \leq h_2(W') \leq \cdots \leq h_n(W')$. Thus $\{h_i(W')\}$ is also a feasible solution of (5). Denote $\Gamma = \{x \geq 0 | \sum_{i=1}^{n} p_i x_i = W; x_{q+1}, \cdots, x_n \geq K\}$ the feasible solution set, and denote Ω the feasible solution set of (5), i.e., $\Omega = \{x \geq 0 | \sum_{i=1}^{n} p_i x_i = W; 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n, x_{q+1} \geq K\}$. It is obvious that $\Omega \subset \Gamma$. We claim that $\{h_i(W')\}$ is an optimal solution of (5). In fact, since $\Omega \subset \Gamma$, we have $V(\Gamma) \geq V(\Omega)$. Here $V(\Gamma)$ and $V(\Omega)$ are the optimal objective function values. As $\{h_i(W')\} \in \Omega$, we have $V(\Omega) \geq \sum_{i=1}^{n} \pi_i u(h_i(W')) = V(\Gamma)$. This indicates $\{h_i(W')\}$ is the optimal solution of (5) too. $\square$

4 Financial Innovation

In addition to the three traditional uses of financial innovation discussed in the Introduction, we identify a new application which in general enhances welfare, facilitates the characterization of the optimal portfolio allocation, and especially reduces complexity. As noted in Section 2, the Value–at–Risk constrained utility maximization Problem 1 is NP–hard. However, if the reverse order Condition 1 holds, Problem 1 can be solved in polynomial time. This suggests that if we are able to suitably transform the state space to satisfy the reverse order condition, then the transformed Problem 1 will have a polynomial time solution.

4.1 State Splitting

Consider an economy where the original state space can be augmented by splitting a state into two or more states. State splitting can be achieved by applying a public randomization devise when state i arises (such as a roll of dice). This augments the original state space.

Definition 3. State splitting by an integer factor γ_i entails dividing state i into γ_i equally likely sub–states, each of which occur with probability $\pi_i^a = \pi_i/\gamma_i$ and cost $p_i^a = p_i/\gamma_i$.

Example 1 (continued). We now relate Examples 1 and 3. State 1 in Example 1 can be transformed into states $1A$ and $1B$ of Example 3 by splitting state 1 in half so that one ends up with $\pi_{1A} = \pi_{1B} = \pi_2 = 1/3$, while the prices are respectively $p_{1A} = p_{1B} = 1/6$, $p_2 = 1/4$. Thus state splitting for the Example 1 configuration, turns this case into Example 3.

Claim 1. *The procedure of state splitting weakly enhances the investor's expected utility.*

This claim follows directly from the fact that any portfolio that was feasible prior to state splitting, remains feasible.

We already showed that the case of Example 3 preserves the monotonicity of the solution with respect to the state price density and raises the expected utility. Denote the common denominator of the probabilities of the original state space $\{\pi_i | i = 1, .., n\}$ by τ. This is always possible if e.g. the probabilities are rationals.

The following is directly implied by the discussion in Section 3.2.

Claim 2. *Suppose the state probabilities $\{\pi_i | i = 1, \ldots, n\}$ have a common denominator $1/\tau$ for some integer τ. Then each state i can be split by factor $\gamma_i = \pi_i \tau$, such that the probability distribution function associated with the augmented state space is discrete uniform, $\pi_i^a = \pi^a$, the VaR constrained optimal portfolio allocation is easily characterized since it is monotonic in prices.*

We do not however require a common denominator since it is not necessary to render the state space discrete uniform. In general, only some states have to be adjusted to satisfying the reverse ordering condition from Section 3.3.

Claim 3. *Suppose there exists a vector of integers* $(\gamma_1, \ldots, \gamma_n)$ *such state splitting implies that reverse ordering conditions is satisfied. Then the VaR constrained portfolio optimization problem is in P.*

Proof. Apply Theorem 3. □

Since the expected utility of the risk constrained investor's optimal portfolio rises weakly in response to state splitting (as observed by the fact that the Lagrange multiplier associated with the VaR constraint declines), investors are in fact stimulated to financial innovation (state splitting) by the risk regulation itself. Hence, we conclude that the probabilistic type of downward risk constraints provides incentives to financial innovation.

One question unanswered by this theoretical analysis is whether state splitting is realistic. To address this issue, we offer two observations. First, note that the state splitting procedure is somewhat analogous to the concept of mixed strategy solutions in game theory, though here the randomization is always weakly welfare improving. Second, we consider a class of securities that split states. Many European government bonds are randomly callable. A detailed example for the case of Swedish government bonds is given in Green and Rydqvist (1997, 1999). For these so–called lottery bonds, each year a predetermined (at the date of issue) percentage of the bond series is retired early. An annual lottery determines which particular bonds are retired. This lottery thus introduces exogenous randomness into the economy and effectively creates new states. The prediction suggested by this paper is that the demand for such securities will increase due to the imposition of the VaR constraints by the BIS regulations. We show this by revisiting previous examples.

Example 4. We demonstrate the effect of state splitting for the case of the Example 1 portfolio problem. Recall Example 3, and note that since the probabilities are uniform, it splits in effect the Example 1 states in such a way that the reverse ordering condition holds. The unconstrained equilibrium of the Example 1 portfolio is on the MN—segment in Figure 3, while the suboptimal VaR constrained portfolio is at N and the optimal VaR constrained portfolio is at M in terms of Figure 3. Due to state splitting, the calculations in Example 3 indicate it is optimal to move from M in Figure

3 to the position T exactly half way between Q and N on the line segment QN of Figure 3.

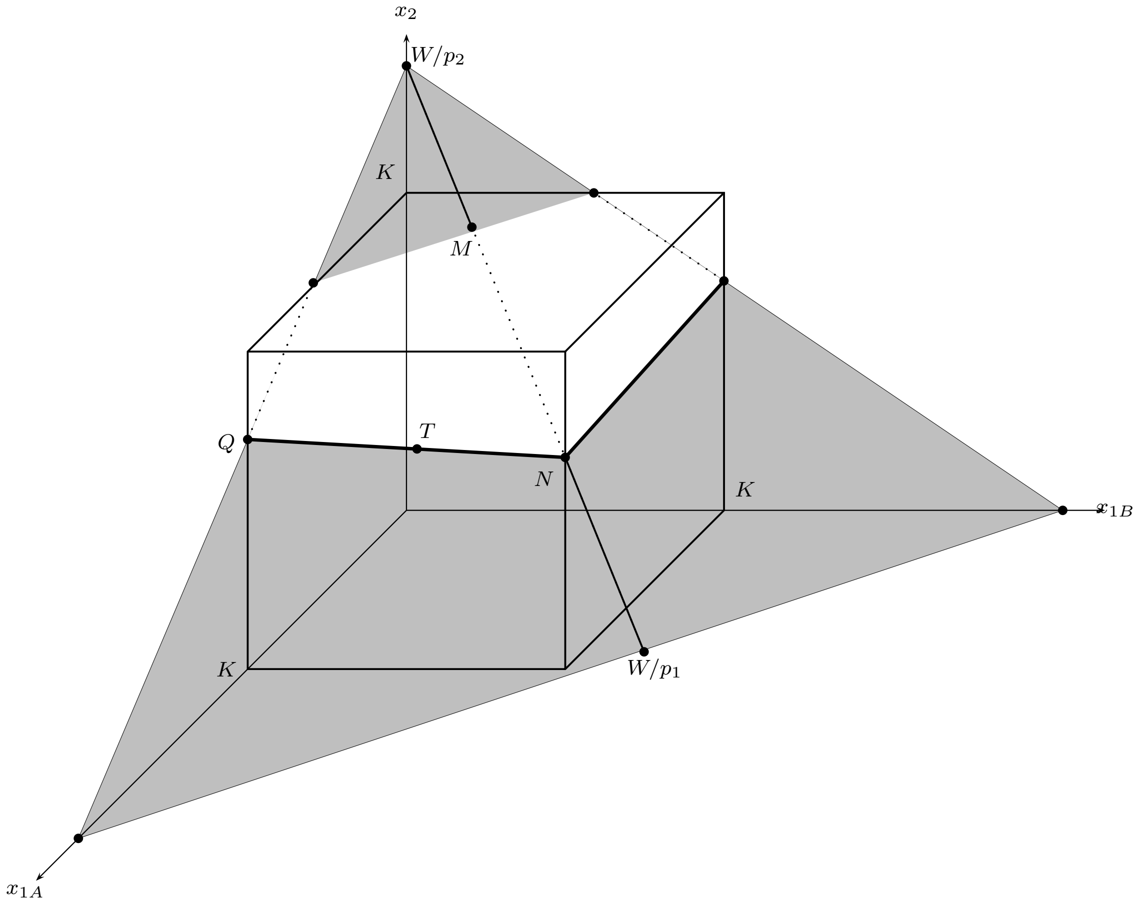

Figure 3: Impact of state splitting
The cube is the VaR constraint, while the gray plane is the budget constraint.

We conclude by illustrating that the reverse ordering condition is not necessary for the optimal VaR constrained portfolio to be monotonic in the state price density.

Example 5. Continuing with Example 2, this problem clearly has too many states to allow depicting geometrically as we do in Figures 1 and 3. We have already seen that the optimal VaR constrained solution, in the absence of state splitting, is not monotonic with regards to the state price density.

Suppose we split state 2 into two different states $2A$ and $2B$ with probabilities $\pi_{2A} = 48/100$ and $\pi_{2B} = 2/100$ and prices $p_{2A} = 48/400$ and $p_{2B} = 2/400$ respectively. The optimal VaR constrained portfolio after state splitting entails $(x_1 \approx 1.11, x_{2A} = 2.21, x_{2B} \approx 2.50, x_3 \approx 3.32, x_4 \approx 4.43)$ which increases the expected utility to $EU = \sum_i \pi_i u(x_i) \approx -0.3511$. Indeed, this portfolio is monotonic with respect to the state price density, yet the reverse ordering condition does not hold. However, one could split such that all states are equally probably with $\pi_i = 1/100$ and scale prices accordingly. From this, optimality is established.

5 Conclusion

In recent years, Value–at–Risk has become the most common downside risk measure in the financial industry, underpinning both internal risk management systems and the supervisory systems. We investigate optimal portfolio construction under risk constraints. The analysis of deterministic downside risk constraints, e.g. portfolio insurance, is well known in the literature. In contrast, VaR, which can be considered as probabilistic portfolio insurance, has received scant attention. We demonstrate that the general VaR constraint turns the optimal portfolio selection problem into a complex problem, belonging in the class of NP–hard.

We subsequently provide a sufficient condition, such that the portfolio selection problem is no longer NP–hard, and demonstrate that investors have an incentive for financial innovation by the introduction of new securities, even though markets are initially complete. The new securities randomize within existing primitive securities and effectively allow investors to better approximate the VaR constraint and thus improve their expected utility. Furthermore, these securities enable polynomial time solutions. This application of financial innovation has not been considered in the extant literature.

6 Proofs

Proof of Theorem 2. We show that Problem 1 can be transformed into a simple problem if π and p satisfy the reverse order relation, that is, $\pi_i \leq \pi_j$ if and only if $p_i \geq p_j$. First we claim that there exists an optimal solution x^* of Problem 1 such that $0 \leq x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ if (π, p) satisfies Condition 1.

For each feasible solution x of Problem 1 , we denote $(x[1], x[2], \cdots, x[n])$ be a permutation of $(1, 2, \cdots, n)$ such that $x_{x[1]} \leq x_{x[2]} \leq \cdots \leq x_{x[n]}$.

Let x^* be an optimal solution of Problem 1. Let j be the smallest subscript such that $\pi_{x^*[j]} \geq \pi_{x^*[j+1]}$ and $p_{x^*[j]} \leq p_{x^*[j+1]}$, and at least one of those two inequalities is strict. Let

$$y = \frac{p_{x^*[j]} x^*_{x^*[j]} + p_{x^*[j+1]} x^*_{x^*[j+1]}}{p_{x^*[j]} + p_{x^*[j+1]}}$$

and

$$z = \frac{p_{x^*[j]} x^*_{x^*[j]} + p_{x^*[j+1]} x^*_{x^*[j+1]} - p_{x^*[j]} K}{p_{x^*[j+1]}}.$$

Let us construct a new solution x^+ such that

$$\begin{aligned} & x^+[j] = x^*[j+1],\ x^+[j+1] = x^*[j]; \\ & x^+[i] = x^*[i], \quad i \neq j, j+1 \end{aligned}$$

$$\begin{array}{ll} x^+_{x^+[j]} = z, x^+_{x^+[j+1]} = K & \text{if } x^*_{x^*[j]} < K \leq x^*_{x^*[j+1]} \text{ and } y < K \\ x^+_{x^+[j]} = y, x^+_{x^+[j+1]} = y & \text{otherwise} \\ x^+_{x^+[i]} = x^*_{x^*[i]}, & i \neq j, j+1 \end{array}$$

In the sequel, we show that such a x^+ is an optimal solution of Problem 1 as well.

We can easily see that such defined x^+ satisfies the budget constraint.

To prove the optimality of x^+, we need show that x^+ satisfies the order condition and insurance constraint, and the objective function value of x^+ is not less than the value of x^*. To this end, we consider three cases separately:

Case 1: $x^*_{x^*[j]} < K \leq x^*_{x^*[j+1]}$ and $y < K$.

For this case, we have defined $x^+_{x^+[j]} = z, x^+_{x^+[j+1]} = K$. From the definition, we can easily calculate that $z < K \leq x^*_{x^*[j+1]}$.

Since $p_{x^*[j]} \leq p_{x^*[j+1]}$ and $x^*_{x^*[j]} < K$, we have

$$(p_{x^*[j]} - p_{x^*[j+1]})(x^*_{x^*[j]} - K) \geq 0 \tag{9}$$

From (9), we can obtain

$$x^*_{x^*[j+1]} + x^*_{x^*[j]} \leq K + z \tag{10}$$

Let

$$\lambda_1 = \frac{x^*_{x^*[j+1]} - K}{x^*_{x^*[j+1]} - x^*_{x^*[j]}},$$

and

$$\lambda_2 = \frac{x^*_{x^*[j+1]} - z}{x^*_{x^*[j+1]} - x^*_{x^*[j]}}.$$

From (10), we have

(i) $x^*_{x^*[j]} \leq z$, hence the order condition $x^+_{x^+[1]} \leq x^+_{x^+[2]} \leq \cdots \leq x^+_{x^+[n]}$ follows from $x^*_{x^*[1]} \leq x^*_{x^*[2]} \leq \cdots \leq x^*_{x^*[n]}$. Further we can easily see that $\sum\limits_{x_i^+ \geq K} \pi_i \geq \sum\limits_{x_i^* \geq K} \pi_i \geq 1 - \delta$ which follows from $\pi_{x^+[j]} = \pi_{x^*[j+1]} \leq \pi_{x^*[j]} = \pi_{x^+[j+1]}$. This indicates that x^+ satisfies the insurance constraint since x^* satisfies the insurance constraint.

(ii) $\lambda_1 + \lambda_2 \leq 1$. Combine with $u(x^*_{x^*[j]}) \leq u(x^*_{x^*[j+1]})$, we have

$$(1 - \lambda_1)(u(x^*_{x^*[j+1]}) - u(x^*_{x^*[j]})) \geq \lambda_2(u(x^*_{x^*[j+1]}) - u(x^*_{x^*[j]})) \tag{11}$$

From the concavity of $u(x)$, we also have

$$u(K) \geq \lambda_1 u(x^*_{x^*[j]}) + (1 - \lambda_1)u(x^*_{x^*[j+1]}) \tag{12}$$
$$u(z) \geq \lambda_2 u(x^*_{x^*[j]}) + (1 - \lambda_2)u(x^*_{x^*[j+1]}) \tag{13}$$

Combine (11) with (12) and (13), we can get

$$u(K) - u(x^*_{x^*[j]}) \geq u(x^*_{x^*[j+1]}) - u(z) \tag{14}$$

Together with $\pi_{x^*[j]} \geq \pi_{x^*[j+1]}$, we obtain

$$\begin{aligned}\pi_{x^*[j]} u(x^*_{x^*[j]}) + \pi_{x^*[j+1]} u(x^*_{x^*[j+1]}) &\leq \pi_{x^*[j]})u(K) + \pi_{x^*[j+1} u(z) \\ &= \pi_{x^+[j]} u(x^+_{x^+[j]}) + \pi_{x^+[j+1]} u(x^+_{x^+[j+1]})\end{aligned} \tag{15}$$

Case 2: $x^*_{x^*[j]} \geq K$ or $x^*_{x^*[j+1]} < K$.

In this case, we have defined $x^+_{x^+[j]} = x^+_{x^+[j+1]} = y$.

First we have

$$x^*_{x^*[j]} \leq y \leq x^*_{x^*[j+1]}$$

Hence the order condition remains true. Furthermore we have

$$\sum_{x_i^+ \geq K} \pi_i = \sum_{x_i^* \geq K} \pi_i \geq 1 - \delta,$$

the insurance constraint is satisfied.

Next we like to show x^+ remains optimal. For this purpose, we claim that

$$\sum_{i=1}^n \pi_i u(x_i^+) \geq \sum_{i=1}^n \pi_i u(x_i^*).$$

Notice that

$$\begin{aligned}
&\sum_{i=1}^n \pi_i u(x_i^+) - \sum_{i=1}^n \pi_i u(x_i^*) \\
&= (\pi_{x^*[j]} + \pi_{x^*[j+1]}) u(\tfrac{p_{x^*[j]} x^*_{x^*[j]} + p_{x^*[j+1]} x^*_{x^*[j+1]}}{p_{x^*[j]} + p_{x^*[j+1]}}) \\
&\quad -(\pi_{x^*[j]} u(x^*_{x^*[j]}) + \pi_{x^*[j+1]} u(x^*_{x^*[j+1]})),
\end{aligned}$$

we only need to show that

$$\begin{aligned}
&(\pi_{x^*[j]} + \pi_{x^*[j+1]}) u(\tfrac{p_{x^*[j]} x^*_{x^*[j]} + p_{x^*[j+1]} x^*_{x^*[j+1]}}{p_{x^*[j]} + p_{x^*[j+1]}}) \\
&\geq \pi_{x^*[j]} u(x^*_{x^*[j]}) + \pi_{x^*[j+1]} u(x^*_{x^*[j+1]})
\end{aligned}$$

Denote $\lambda = \frac{p_{x^*[j]}}{p_{x^*[j]} + p_{x^*[j+1]}}$. Then $0 < \lambda < 1$.

From $\pi_{x^*[j]} \geq \pi_{x^*[j+1]}$, $p_{x^*[j]} \leq p_{x^*[j+1]}$, we have $\pi_{x^*[j]} p_{x^*[j+1]} \geq \pi_{x^*[j+1]} p_{x^*[j]}$. Furthermore we can deduce that

$$p_{x^*[j]}(\pi_{x^*[j]} + \pi_{x^*[j+1]}) - (p_{x^*[j]} + p_{x^*[j+1]}) \pi_{x^*[j]} \leq 0,$$

or equivalently

$$\lambda(\pi_{x^*[j]} + \pi_{x^*[j+1]}) - \pi_{x^*[j]} \leq 0 \tag{16}$$

From the monotone property of $u(x)$, we have $u(x^*_{x^*[j]}) - u(x^*_{x^*[j+1]}) \leq 0$. Combine with (16), it follows that

$$[\lambda(\pi_{x^*[j]} + \pi_{x^*[j+1]}) - \pi_{x^*[j]}](u(x^*_{x^*[j]}) - u(x^*_{x^*[j+1]})) \geq 0$$

This inequality is equivalent to

$$\begin{aligned}(\pi_{x^*[j]} + \pi_{x^*[j+1]})[\lambda u(x^*_{x^*[j]}) + (1-\lambda)u(x^*_{x^*[j+1]})] \\ \geq \pi_{x^*[j]}u(x^*_{x^*[j]}) + \pi_{x^*[j+1]}u(x^*_{x^*[j+1]})\end{aligned} \tag{17}$$

Notice that $x^+_{x^+[j]} = x^+_{x^+[j+1]} = \lambda x^*_{x^*[j]} + (1-\lambda)x^*_{x^*[j+1]}$, from the concavity of $u(x)$, we can obtain that

$$u(x^+_{x^+[j]}) = u(x^+_{x^+[j+1]}) \geq \lambda u(x^*_{x^*[j]}) + (1-\lambda)u(x^*_{x^*[j+1]}) \tag{18}$$

Combine (17) and (18), to get

$$(\pi_{x^*[j]} + \pi_{x^*[j+1]})u(x^+_{x^+[j]}) \geq \pi_{x^*[j]}u(x^*_{x^*[j]}) + \pi_{x^*[j+1]}u(x^*_{x^*[j+1]})$$

Hence we conclude that x^+ is also an optimal solution of Problem 1.

Case 3: $x^*_{x^*[j]} < K \leq x^*_{x^*[j+1]}$ but $y \geq K$.

In this case, we have also defined $x^+_{x^+[j]} = x^+_{x^+[j+1]} = y$.

First we have $\sum\limits_{x^+_i \geq K} \pi_i = \sum\limits_{x^*_i \geq K} \pi_i + \pi_{x^*[j]} \geq 1-\delta$, thus the insurance constraint is satisfied.

Following same steps as Case 2, we can show that the order condition remains true, and the objective function value of x^+ is not worse than the objective function value of x^*.

Therefore combining the three cases, we can conclude that there exists an optimal solution x^+ such that $\pi_{x^+[j]} \leq \pi_{x^+[j+1]}$ and $p_{x^+[j]} \geq p_{x^+[j+1]}$.

Using above technique repeatedly, we can always construct an optimal solution of which the order of entries is in the order of π_i, if Problem 1 has an optimal solution.

This indicates that there is an optimal solution x^* such that

$$0 \leq x^*_1 \leq x^*_2 \leq \cdots \leq x^*_n$$

if (π, p) satisfies Condition 1.

Let q be the number determined by (4). We show that $x^*_{q+1} \geq K$ for such an optimal solution. In fact, we have $\sum\limits_{x^*_i < K} \pi_i \leq \delta$ by the feasibility of x^*. If $x^*_{q+1} < K$, we get

$$\sum_{x^*_i < K} \pi_i \geq \sum_{i=1}^{q+1} \pi_i > \delta$$

from (4). This is a contradiction.

Conversely if x is a solution such that $\sum_{i=1}^{n} p_i x_i = W$, $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, and $x_{q+1} \geq K$, we have $\sum_{x_i \geq K} \pi_i \geq \sum_{i=q+1}^{l} \pi_i \geq 1 - \delta$. x becomes a feasible solution of Problem 1. Hence we can conclude that Problem 1 can be reformulated as (5) when (π, p) satisfies Condition 1. $\square$

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