Comparing downside risk measures for heavy tailed distributions

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Abstract

In this paper we study some prominent downside risk measures for heavy tailed distribution. Using the notion of regular variation to define heavy tailed distributions we provide approximations of the risk measures in the tail region. We show that the downside risk measures produce similar and consistent ranking of risk. However, Expected Shortfall may not always distinguish between the differing risk levels of assets.

KEY WORDS: downside risk measures, heavy tailed distribution, regular variation

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1 Introduction

Downside risk measures may be defined as measures of "distance" between a risky situation and the corresponding risk-free situation when only unfavourable discrepancies contribute to the "risk" (Dhaene et al., 2003). Early literature on downside risk measures dates back to the "safety first" rule of Roy (1952). Subsequently lower partial moments were developed (Bawa, 1975; Fishburn, 1977) where risk was defined as probability weighted functions of deviations below certain target return. There is a renewed interest in downside risk measures in recent times due to the prominence of concepts like Value-at-Risk (VaR) and Expected Shortfall (ES) for financial risk management and prudential regulation.

In this paper we study some important downside risk measures, viz., lower partial moments (of second, first and zeroth orders), VaR and ES for heavy tailed asset returns. Using the notion of "regular variation" to define heavy tailed behaviour, we provide approximations of the risk measures in the tail region.

Further, it is analytically shown that the heavy tailed feature induces similar asset rankings regardless of the particular risk measure being used.

2 Heavy tailed distribution and downside risk measures

A plethora of empirical studies have established that asset returns tend to have heavy tailed distributions (Mandelbrot, 1963; Pagan, 1996; Engle, 1982; Jansen and de Vries, 1991). Heavy tailed distributions are often defined in terms of higher than normal kurtosis. However, the kurtosis of a distribution may be high if either the tails of the cdf are heavier than the normal or if the center is more peaked, or both. Further, it is not only the higher than normal kurtosis, but also failure of higher moments that define heavy tails. In this paper we define heavy tailed distribution as one characterised by the failure of the moments of order m (> 0) or higher. Such distributions have tails that exhibit a power type behaviour like the Pareto distribution, as commonly observed in finance. Such tail behaviour can be mathematically defined by using the notion of "regular variation", as defined below[1] .

2.1 Regular Variation

A cdf $F(x)$ varies regularly at minus infinity with tail index $\alpha > 0$ if

$$\lim_{t\to\infty} \frac{F(-tx)}{F(-t)} = x^{-\alpha} \quad \forall x > 0 \tag{1}$$

An implication of regular variation is that, to a first order approximation, all distributions have a tail comparable to the Pareto distribution:

$$F(-x) = Ax^{-\alpha}[1 + o(1)], \; x > 0, \quad \text{for } \alpha > 0 \text{ and } A > 0 \tag{2}$$

[1]For an encyclopedic treatment of regular variation, see Bingham et al. (1987); Resnick (1987).

Further, to a second order approximation, the tail of a regularly varying cdf can be written as

$$F(-x) = Ax^{-\alpha}\left[1 + Bx^{-\beta} + O\left(x^{-\beta}\right)\right], \text{ as } x \to \infty \text{ for } \alpha > 2,\ \beta > 0,\ A > 0 \text{and } B \neq 0 \quad (3)$$

For distributions with regularly varying tails, moments of order $m > \alpha$ are unbounded and therefore these distributions display heavy tailed behaviour. The power α is called the tail index and it determines the number of bounded moments. It is readily verified that Student–t distributions, among others, vary regularly at infinity, has degrees of freedom equal to the tail index and satisfies the above approximation. Likewise, the stationary distribution of the popular GARCH(1,1) process has regularly varying tails, see de Haan et al. (1989).

3 Downside risk measures for heavy-tailed distribution

In this section we discuss how the downside risk measures can be formed under the assumption of regularly varying tails. Also we discuss whether the downside risk measures provide consistent preference ordering under the condition of regularly varying tails.

Before proceeding, we provide the definitions of the risk measures under consideration below.

1. Second Lower Partial Moment or Semi-variance (SLPM)

$$SLPM = \int_{-\infty}^{q} (q-x)^2 f(x)dx \quad = \quad 2\int_{-\infty}^{q} (q-x)F(x)dx$$

2. First Lower Partial Moment (FLPM)

$$FLPM = \int_{-\infty}^{q} (q-x) f(x)dx \quad = \quad \int_{-\infty}^{q} F(x)dx$$

3. Zeroth Lower Partial Moment (ZLPM)

$$ZLPM = \int_{-\infty}^{q} f(x)dx = F(q)$$

4. Value-at-Risk (VaR): If $F(q)$ is fixed at p, then the inverse of ZLPM gives Value-at-Risk (VaR) as

$$VaR_p = -F^{-1}(p) = -q$$

   VaR is defined as the maximum potential loss to an investment with a pre-specified confidence level $(1-p)$.

5. Expected Shortfall (ES): When the return distribution is continuous, ES at confidence

level $(1-p)$ is defined as

$$\begin{aligned} \text{ES}_p &= E(x|x \leq VaR_p) \\ &= \int_{-\infty}^{q} x \ \frac{f(x)}{F(q)} dx \\ &= q - \frac{1}{F(q)} \ \ FLPM \end{aligned}$$

In the following propositions we provide expressions for approximating the risk measures.

Proposition 1 *If the asset return distribution is heavy tailed with tail index $\alpha > 0$ and tail coefficient $A > 0$, then the downside risk measures can be approximated as follows:*

1. $\text{SLPM}(q) \approx \frac{2Aq^{-\alpha+2}}{(1-\alpha)(2-\alpha)}, \ \alpha \neq 1, \ \alpha \neq 2$
2. $\text{FLPM}(q) \approx \frac{Aq^{-\alpha+1}}{(1-\alpha)}, \ \alpha \neq 1$
3. $\text{ZLPM}(q) \approx Aq^{-\alpha}, \ \alpha > 0$
4. $VaR(p) \approx \left(\frac{A}{p}\right)^{\frac{1}{\alpha}}$
5. $\text{ES}(q) \approx \frac{\alpha q}{(\alpha-1)}$

Proof:
See Appendix A.

Proposition 1 provides approximations for the downside risk measures in terms of the parameters of the tail approximation of heavy tailed distribution. If the tail index $\alpha = 2$ or $\alpha = 1$ then the risk measure SLPM is undefined. Similarly if $\alpha = 1$ FLPM is undefined. However $\alpha = 1$ is a rare situation in finance, as it would imply unbounded mean.

Proposition 2 *If the asset X has a regularly varying tail that satisfy the second order approximation (3), then the downside risk measures can be approximated as follows:*

1. $\text{SLPM}(q) \approx 2\left[\frac{Aq^{-\alpha+2}}{(1-\alpha)(2-\alpha)} + \frac{ABq^{-\alpha-\beta+2}}{(\alpha+\beta-1)(\alpha+\beta-2)}\right]$
2. $\text{FLPM}(q) \approx -\frac{Aq^{-\alpha+1}}{\alpha-1}\left[1 + \frac{(\alpha-1)Bq^{-\beta}}{\alpha+\beta-1}\right]$
3. $\text{ZLPM}(q) \approx Aq^{-\alpha}\left[1 + Bq^{-\beta}\right]$
4. $VaR_p(x) \approx A^{\frac{1}{\alpha}} \ p^{-\frac{1}{\alpha}}\left[1 + \frac{B}{\alpha}A^{-\frac{\beta}{\alpha}} \ p^{\frac{\beta}{\alpha}}\right]$
5. $\text{ES}(q) \approx q \ \ \frac{\frac{\alpha}{\alpha-1} + \frac{\alpha+\beta}{\alpha+\beta-1} \ Bq^{-\beta}}{1+Bq^{-\beta}}$

Proof:
See Appendix B.

From Propositions 1 and 2 it is interesting to note that except for ES all the downside risk measures are functions of both the tail coefficient A and the tail index α, while the expression for ES involves only the tail index α and not the tail coefficient A. This can be seen as a drawback of ES since ES will yield the same risk measure for two assets having same tail index but different tail coefficients. Other measures, as seen from the above expressions, take into account both the tail index and the tail coefficients. This is more relevant because financial returns are often found to be similar with respect to the tail index, but vary widely with respect to the tail coefficient.[2]

3.1 Comparing the risk measures under the condition of regular variation

Suppose that asset returns X and Y have regularly varying tails with tail indexes α_1 and α_2 and tail coefficients A_1 and A_2 respectively.

Proposition 3 *If $\alpha_1 > \alpha_2$ and $A_1 = A_2$, then the following relationships hold.*

1. $\text{SLPM}_x(q) < \text{SLPM}_y(q)$ *for* $\alpha_1 > 2, \ \alpha_2 > 2$
2. $\text{FLPM}_x(q) < \text{FLPM}_y(q)$ *for* $\alpha_1 > 1, \ \alpha_2 > 1$
3. $\text{ZLPM}_x(q) < \text{ZLPM}_y(q)$
4. $VaR_x(p) < VaR_y(p)$
5. $\text{ES}_x(q) < \text{ES}_y(q)$

Proof:
Differentiating the expressions for the downside risk measures as in Proposition 1, it follows that each downside risk measure is decreasing in α, for large $q > 1$. Hence the result.

Thus, in the tail region, we can order X and Y in a clear manner with respect to each of the downside risk measures. The ordering is consistent with the assumption that X is less risky than Y. Thus, far in the tail region, all measures provide similar preference ranking. This is in line with the recent empirical findings of Hahn et al. (2002). Using real world data from the trading book of an investment bank, Hahn et al. (2002) found empirically that many of the downside risk measures (including the ones considered in this paper) assess risk of the trading portfolios in nearly the same way. Proposition 3 explain this similarity in an analytical manner.

Now, assume that $\alpha_1 = \alpha_2$ but the tail coefficients $A_1 \neq A_2$. Without loss of generality, let $A_1 < A_2$. In this case we have $F_x(-x) < F_y(-x)$, so that the tail of the distribution of Y is heavier than that of X. Thus, although the number of bounded moments are same for X and Y, Y has a fatter tail than X, and hence more risky.

Proposition 4 *If $\alpha_1 = \alpha_2$ but $A_1 < A_2$, then the following relationships hold.*

1. $\text{SLPM}_x(q) < \text{SLPM}_y(q)$
2. $\text{FLPM}_x(q) < \text{FLPM}_y(q)$
3. $\text{ZLPM}_x(q) < \text{ZLPM}_y(q)$
4. $VaR_x(p) < VaR_y(p)$
5. $\text{ES}_x(q) = \text{ES}_y(q)$

[2]See, eg.,Hyung and de Vries (2002).

Proof:
Above can be proved easily by using the results from the Proposition 1.

Thus, all the risk measures, except ES give consistent ordering of the assets X and Y in this case. However, ES does not differentiate between the risk levels of X and Y by ignoring the scale coefficients.

4 Conclusion

We examine downside risk measures for heavy tailed distributions defined as distributions with regularly varying tails. Using tail approximations of regularly varying tails, we provide expressions that approximate the various downside risk measures as functions of the tail coefficient and tail index. We show that when two heavy tailed distributions have different tail indexes but same tail coefficients, all downside risk measures provide consistent preference ordering as under the notion of risk conveyed by the tail indexes. In this case the measures SLPM and FLPM do not display any clear preference order. If the tail indexes are same but the tail coefficients vary, then all but ES provide consistent preference ordering as under the risk information conveyed by the differing tail coefficients. In this case, ES estimates risk equally, thus ignoring the different risk levels of the assets arising out of different tail coefficients.

Appendix A: Derivation of the expressions in Proposition 1

Suppose that the distribution of X has a regularly varying tail. Then, to a fist order approximation,

$$F(-x) \approx A\ x^{-\alpha},\ \text{as } x \to \infty \text{ where } A > 0,\ \alpha > 0$$

1. SLPM

$$\begin{aligned}
\text{SLPM}(q) &= 2\int_{-\infty}^{q} (q-x)\ F(x)dx \\
&\approx 2\int_{-\infty}^{q} (q-x)\ Ax^{-\alpha}dx \\
&= 2\int_{-\infty}^{q} qAx^{-\alpha}dx - 2\int_{-\infty}^{q} Ax^{-\alpha+1}dx \\
&= 2qA\left|\frac{x^{-\alpha+1}}{-\alpha+1}\right|_{-\infty}^{q} - 2A\left|\frac{x^{-\alpha+2}}{-\alpha+2}\right|_{-\infty}^{q},\ \alpha > 2 \\
&= 2A\left[\frac{q^{-\alpha+2}}{-\alpha+1} - \frac{q^{-\alpha+2}}{-\alpha+2}\right] \\
&= \frac{2Aq^{-\alpha+2}}{(1-\alpha)(2-\alpha)}
\end{aligned}$$

2. FLPM(q)

$$\begin{aligned}
\text{FLPM}(q) &= \int_{-\infty}^{q} F(x)dx \\
&\approx \int_{-\infty}^{q} Ax^{-\alpha}dx \\
&= A\left|\frac{x^{-\alpha+1}}{-\alpha+1}\right|_{-\infty}^{q},\ \alpha > 1 \\
&= \frac{Aq^{-\alpha+1}}{1-\alpha}
\end{aligned}$$

3. ZLPM(q)

$$\begin{aligned}
\text{ZLPM}(q) &= F(q) \\
&\approx Aq^{-\alpha}
\end{aligned}$$

4. $VaR_p(x)$

$$\begin{aligned}
F\left(VaR_p(x)\right) &= p \\
p &\approx A\left[VaR_p(x)\right]^{-\alpha} \\
VaR_p(x) &\approx \left(\frac{A}{p}\right)^{\frac{1}{\alpha}}
\end{aligned}$$

5. ES(q)

$$
\begin{aligned}
\mathrm{ES}(q) &= q - \frac{\mathrm{FLPM}(q)}{F(q)} \\
&\approx q - \frac{Aq^{-\alpha+1}}{Aq^{-\alpha}\,(1-\alpha)} \\
&= \frac{\alpha\, q}{(\alpha-1)}
\end{aligned}
$$

Appendix B: Derivation of the expressions in Proposition 2

Suppose that X has a regularly varying tail that satisfy the following second order approximation as $x \to \infty$

$$
F(-x) \approx Ax^{-\alpha}\left[1 + \ Bx^{-\beta}\right],\ A > 0,\ \alpha > 2,\ \beta > 0 \text{ and } B \neq 0
$$

1. SLPM(q)

$$
\begin{aligned}
\mathrm{SLPM}(q) &= 2\int_{-\infty}^{q} (q-x)\,F(x)dx \\
&= 2q\int_{-\infty}^{q} F(x)dx - 2\int_{-\infty}^{q} xF(x)dx \\
&= T_1 - T_2,\ \text{say}
\end{aligned}
$$

Using the second order tail approximation, we have

$$
\begin{aligned}
T_1 &\approx 2q\int_{-\infty}^{q} \left[Ax^{-\alpha} + ABx^{-\alpha-\beta}\right] dx \\
&= 2q\left|\frac{Ax^{-\alpha+1}}{-\alpha+1} + \frac{ABx^{-\alpha-\beta+1}}{-\alpha-\beta+1}\right|_{-\infty}^{q} \\
&= 2q\left[\frac{Aq^{-\alpha+1}}{-\alpha+1} + \frac{ABq^{-\alpha-\beta+1}}{-\alpha-\beta+1}\right],\ \alpha+\beta > 1 \\
T_2 &\approx 2\int_{-\infty}^{q} \left[Ax^{-\alpha+1} + ABx^{-\alpha-\beta+1}\right] dx \\
&= 2\left[\frac{Aq^{-\alpha+2}}{-\alpha+2} + \frac{ABq^{-\alpha-\beta+2}}{-\alpha-\beta+2}\right],\ \text{provided } \alpha+\beta > 2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{SLPM}(q) &\approx 2\left[\frac{Aq^{-\alpha+2}}{-\alpha+1} - \frac{ABq^{-\alpha-\beta+2}}{\alpha+\beta-1} - \frac{Aq^{-\alpha+2}}{-\alpha+2} + \frac{ABq^{-\alpha-\beta+2}}{\alpha+\beta-2}\right] \\
&= 2\left[\frac{Aq^{-\alpha+2}}{(1-\alpha)(2-\alpha)} \ + \ \frac{ABq^{-\alpha-\beta+2}}{(\alpha+\beta-1)(\alpha+\beta-2)}\right]
\end{aligned}
$$

2. FLPM(q)

$$\begin{aligned}
\text{FLPM}(q) &= \int_{-\infty}^{q} F(x)dx \\
&\approx \int_{-\infty}^{q} \left[Ax^{-\alpha} + ABx^{-\alpha-\beta}\right] dx \\
&= \frac{Aq^{-\alpha+1}}{-\alpha+1} + \frac{ABq^{-\alpha-\beta+1}}{-\alpha-\beta+1},\ \alpha+\beta > 1 \\
&= -\frac{Aq^{-\alpha+1}}{\alpha-1}\left[1 + \frac{(\alpha-1)Bq^{-\beta}}{\alpha+\beta-1}\right]
\end{aligned}$$

3. ZLPM(q)

$$\begin{aligned}
\text{ZLPM}(q) &= F(q) \\
&\approx Aq^{-\alpha}\left[1 + Bq^{-\beta}\right]
\end{aligned}$$

4. ES(q)

$$\begin{aligned}
\text{ES}(q) &= q - \frac{\text{FLPM}(q)}{F(q)} \\
&\approx q - \frac{-\frac{Aq^{-\alpha+1}}{\alpha-1}\left[1 + \frac{(\alpha-1)Bq^{-\beta}}{\alpha+\beta-1}\right]}{Aq^{-\alpha}\left[1 + Bq^{-\beta}\right]} \\
&= \frac{Aq^{-\alpha+1}\left[1 + Bq^{-\beta}\right] \ + \ \frac{Aq^{-\alpha+1}}{\alpha-1}\left[1 + \frac{(\alpha-1)Bq^{-\beta}}{\alpha+\beta-1}\right]}{Aq^{-\alpha}\left[1 + Bq^{-\beta}\right]} \\
&= q\ \frac{1 + Bq^{-\beta} + \ \frac{1}{\alpha-1}\ + \frac{Bq^{-\beta}}{\alpha+\beta-1}}{1 + Bq^{-\beta}} \\
&= q\ \frac{\frac{\alpha}{\alpha-1}\ + \frac{\alpha+\beta}{\alpha+\beta-1}\ Bq^{-\beta}}{1 + Bq^{-\beta}}
\end{aligned}$$

5. $VaR_p(x) = F^{-1}(p)$. We use Bruijn's theory of asymptotic inversion (Bingham et al., 1987) (pages 28-29) to establish this result.

Let $p = F(x)$. Using the second order approximation of $F(x)$,

$$\begin{aligned}
p &\approx Ax^{-\alpha}\left[1 + Bx^{-\beta}\right] \\
\frac{p}{A} &\approx x^{-\alpha}\left[1 + Bx^{-\beta}\right] \\
y &\approx x^{-\alpha}\left[1 + Bx^{-\beta}\right] \text{ where } y = \frac{p}{A} && (4) \\
x &\approx y^{-\frac{1}{\alpha}}\left[1+Bx^{-\beta}\right]^{\frac{1}{\alpha}} \\
&= y^{-\frac{1}{\alpha}} g\left(x(y)\right) \\
&= y^{-\frac{1}{\alpha}} + y^{-\frac{1}{\alpha}} g\left(x(y)\right) - 1\} \\
&= y^{-\frac{1}{\alpha}} + \epsilon(y) \; say && (5)
\end{aligned}$$

Clearly,

$$\frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}} \rightarrow 0 \text{ as } x \rightarrow \infty \; since, as \; x \rightarrow \infty, g\left(x(y)\right) \rightarrow 1$$

Using (5) in (4), we have the following:

$$\begin{aligned}
y &\approx x^{-\alpha}\left[1 + Bx^{-\beta}\right] \\
&= \left[y^{-\frac{1}{\alpha}} + \epsilon(y)\right]^{-\alpha}\left[1 + \; B\left(y^{-\frac{1}{\alpha}} + \epsilon(y)\right)^{-\beta}\right] \\
&= y\left(1 + \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}\right)^{-\alpha}\left[1 + By^{\frac{\beta}{\alpha}}\left(1 + \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}^{-\beta}\right)\right] \\
1 &\approx \left(1 + \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}\right)^{-\alpha}\left[1 + By^{\frac{\beta}{\alpha}}\left(1 + \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}^{-\beta}\right)\right] && (6)
\end{aligned}$$

Now, let

$$\begin{aligned}
u &= \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}, and \\
f(u) &= (1+u)^{-\alpha}
\end{aligned}$$

Taylor's expansion of $f(u)$ around $u = 0$ gives

$$\begin{aligned}
f(u) &= f(0) + uf^{'}(u)|_{u=0} + \frac{u^2}{2!}f^{''}(u)|_{u=0} + \frac{u^3}{3!}f^{'''}(u)|_{u=0} + ... \\
&= 1 - \alpha u + \alpha(\alpha+1)\frac{u^2}{2} - \alpha(\alpha+1)(\alpha+2)\frac{u^3}{6} + ...
\end{aligned}$$

Thus, to a first order approximation,

$$\begin{aligned} f(u) &\approx 1-\alpha u \\ \left(1+\frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}\right)^{-\alpha} &\approx 1-\alpha\frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}} \end{aligned} \tag{7}$$

Using (7) in (6), we have,to a first order,

$$\begin{aligned} 1 &\approx \left(1-\alpha\ \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}\right)\left[1+By^{\frac{\beta}{\alpha}}\left(1-\beta\ \frac{\epsilon(y)}{y^{-\frac{1}{\alpha}}}\right)\right] \\ &= 1+By^{\frac{\beta}{\alpha}}\left(1-\beta\epsilon(y)\ y^{\frac{1}{\alpha}}\right)-\alpha\ \epsilon(y)\ y^{\frac{1}{\alpha}}-B\ \alpha\ \epsilon(y)\ y^{\frac{\beta}{\alpha}+\frac{1}{\alpha}}\left(1-\beta\epsilon(y)y^{\frac{1}{\alpha}}\right) \\ \alpha\epsilon(y)y^{\frac{1}{\alpha}} &\approx By^{\frac{\beta}{\alpha}}-\epsilon(y)\ y^{\frac{\beta}{\alpha}+\frac{1}{\alpha}}\left(B\beta+B\alpha-B\alpha\beta\epsilon(y)y^{\frac{1}{\alpha}}\right) \\ \epsilon(y) &\approx \frac{B}{\alpha}\ y^{\frac{\beta}{\alpha}-\frac{1}{\alpha}}, \quad \text{to a first order approximation} \end{aligned} \tag{8}$$

Because as $x \to \infty$, the other terms $\to 0$ faster.

Using (8) in (5),

$$\begin{aligned} x &\approx y^{-\frac{1}{\alpha}}+\frac{B}{\alpha}\ y^{\frac{\beta}{\alpha}-\frac{1}{\alpha}} \\ &= y^{-\frac{1}{\alpha}}\left[1+\frac{B}{\alpha}\ y^{\frac{\beta}{\alpha}-\frac{1}{\alpha}}\right] \end{aligned}$$

Substituting $y=\frac{p}{A}$,

$$\begin{aligned} x &\approx A^{\frac{1}{\alpha}}\ p^{-\frac{1}{\alpha}}\left[1+\frac{B}{\alpha}A^{-\frac{\beta}{\alpha}}\ p^{\frac{\beta}{\alpha}}\right] \\ VaR_p(x) &\approx A^{\frac{1}{\alpha}}\ p^{-\frac{1}{\alpha}}\left[1+\frac{B}{\alpha}A^{-\frac{\beta}{\alpha}}\ p^{\frac{\beta}{\alpha}}\right] \end{aligned}$$

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