A numeric method for computing the distribution of a quadratic polynomial of a normal random vector

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Abstract

The paper describes a numeric method for computing an improper integral, representing the distribution function of returns of a complex nonlinear financial portfolio.

1 Introduction

Analytic calculation of a VaR risk measure (value-at-risk) for linear portfolios is a standard procedure, as described e.g. in the RiskMetrics technical document. If nonlinear positions, such as derivative instruments, are present in a portfolio, the linear approximation becomes very poor. In this case nonlinear approximation for portfolio returns is inevitable.

In [1] a method of calculating of the distribution of a quadratic polynomial of a normal random vector was proposed. The method supplies the value of the distribution function as an improper integral. To calculate the integral with given precision one should select the finite range of integration and a numeric method of integration, providing the desired result.

The present paper is devoted to building the method. This method allows optimal selection of integration range and the step of numeric integration, guaranteeing the precision required.

Note that the precision should be concordant with the assumption that the transform of initial normal vector is quadratic. The method may be also extended to other nonlinear transforms for obtaining better approximations.

It is worth noting that Mathematica software package by Wolfram Research [2] contains a method for calculation the distribution function of a quadratic transform of a normal

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random vector; the method uses series expansion, which leads to inappropriate results even for two-dimensional random vector.

2 Quadratic transform

For the sake of independence, in the present section we provide the derivation of expression for the distribution function of transformed vector via parameters of the underlying distribution and parameters of the transform. Recall that χ-squared distribution with ν degrees of freedom and non-centrality parameter δ^2 is a distribution of the random variable

$$(\xi_1 + \delta_1)^2 + \cdots + (\xi_\nu + \delta_\nu)^2,$$

where $\xi_1, \ldots, \xi_\nu$ are independent standard normal variables, and $\delta^2 = \delta_1^2 + \cdots + \delta_\nu^2$.

Let an n-dimensional vector X be joint normal with mean vector $\mu = (\mu_1, \ldots, \mu_n)^T$ and covariance matrix S, which is symmetric and positive definite[3]. Consider a quadratic polynomial $P : \mathbf{R}^n \to \mathbf{R}$ of the form

$$P(x) = x^T C x + b^T x + a, \quad x \in \mathbf{R}^n, \tag{1}$$

where C in a symmetric $n \times n$ matrix, $b \in \mathbf{R}^n$ is a vector, and $a \in \mathbf{R}$. The gradient of P has the form

$$P'(x) = 2Cx + b. \tag{2}$$

Applying this quadratic transform to X we get

$$Y = P(X) = X^T C X + b^T X + a. \tag{3}$$

Now let us derive a formula for distribution function of a random variable Y.

Let $S = A^T A$ be a Choleski decomposition of the matrix S, that is, A is an upper triangular matrix. Consider eigenvalues $\gamma_1, \ldots, \gamma_n$ and eigenvectors of a matrix $R = ACA^T$, which are numbered in such a way, that all zero eigenvalues (if any) would be listed in the end. Denote m the number of nonzero eigenvalues (m equals to the rank of C). Let the columns of a matrix V be the orthonormal eigenvectors of R, so that $V^T = V^{-1}$. Further, let Γ be a diagonal matrix comprised of the eigenvalues of R. In particular , we have $R = V\Gamma V^T$ and $\Gamma = V^T R V$. Denote $D = A^T V$ and consider change of variables $Z = D^{-1}(X - \mu)$, or $X = \mu + DZ$. It is easy to verify that the mean vector of Z equals 0, and its covariance matrix is the unit matrix, that is, the components of Z are independent standard normal random variables. Substituting representation of X via Z into (3), one obtains

$$Y = (\mu^T + Z^T D^T) C (\mu + DZ) + b^T(\mu + DZ) + a = P(\mu) + (P'(\mu))^T DZ + Z^T \Gamma Z$$

[3] In degenerate case one can reduce the dimension of X beforehand, for example, by principal components method, thus obtaining non-degenerate covariance matrix.

$$= Z^T \Gamma Z + d^T Z + P(\mu),$$

where $d^T = (P'(\mu))^T D$. Representing it by components, we have

$$Y = \sum_{k=1}^{m} \left[\gamma_k Z_k^2 + d_k Z_k \right] + \sum_{k=m+1}^{n} d_k Z_k + P(\mu). \tag{4}$$

Now represent the summands of the first sum in (4) as full squares:

$$\gamma_k Z_k^2 + d_k Z_k = \gamma_k \left(Z_k^2 + \frac{d_k}{\gamma_k} Z_k \right) = \gamma_k \left(Z_k + \frac{d_k}{2\gamma_k} \right)^2 - \frac{d_k^2}{4\gamma_k} = \gamma_k Q_k - \frac{d_k^2}{4\gamma_k},$$

where

$$Q_k = \left(Z_k + \frac{d_k}{2\gamma_k} \right)^2$$

is a χ-squared random variable with one degree of freedom and non-centrality parameter

$$\delta_k^2 = \frac{d_k^2}{4\gamma_k^2},$$

$k = 1, \ldots, m$. The rest summands in (4) represent a normal random variable. Collecting non-random summands in (4), we get

$$Y = \sum_{k=1}^{m} \gamma_k Q_k + \beta Q_0 + \alpha, \tag{5}$$

where Q_0 is a standard normal random variable,

$$\beta = \left(\sum_{k=m+1}^{n} d_k^2 \right)^2, \quad \alpha = \mu^T C \mu + b^T \mu + a - \sum_{k=1}^{m} \frac{d_k^2}{4\gamma_k}.$$

It was shown in [1] that the distrubution function of Y is

$$F(y) = \int_0^\infty f(w, y)\, dw, \tag{6}$$

where

$$f(w, y) = \frac{e^{A(w)} \sin(B(w, y) + C(w))}{D(w)}, \tag{7}$$

$$A(w) = -\frac{w^2}{2} \left(\beta^2 + \sum_{k=1}^{m} \frac{4\gamma_k^2 \delta_k^2}{1 + 4\gamma_k^2 w^2} \right), \tag{8}$$

$$B(w, y) = w \left(\alpha - y + \sum_{k=1}^{m} \frac{\gamma_k \delta_k^2}{1 + 4\gamma_k^2 w^2} \right), \tag{9}$$

$$C(w) = \frac{1}{2} \sum_{k=1}^{m} \arctan(2\gamma_k w), \tag{10}$$

$$D(w) = w \left[\prod_{k=1}^{m} \left(1 + 4\gamma_k^2 w^2 \right) \right]^{1/4}. \tag{11}$$

Now let us calculate the integral in (6).

3 Approximate calculation of the integral

The two problems arise in the calculation of the integral (6). First, the integrand (7) in (6) possesses the two singularities: $0/0$ at $w = 0$, and unbounded from the right range of integration. Thus we need to find a finite interval $[l, u] \subset [0, \infty)$ without singularities. As it will be shown below, the singularity at 0 can be easily overcome, so only the problem of selecting the upper limit u remains in effect. Second, we need to choose a numeric method of integration over $[0, u]$, which will also contribute some error to the result.

Consider the singularity at 0. Denote $\sim$ the fact of similar behavior of functions as $w \to 0$, that is, $g(w) \sim h(w)$ means $g(w)/h(w) \to 1$ as $w \to 0$. It follows from (8) – (11) that

$$A(w) \sim 0,\ B(w,y) \sim w\left(\alpha - y + \sum_{k=1}^{m} \gamma_k \delta_k^2\right),\ C(w) \sim w\sum_{k=1}^{m}\gamma_k,\ D(w) \sim w,$$

thus

$$f(w,y) \sim w^{-1} \sin\left[w\left(\alpha - y + \sum_{k=1}^{m}\gamma_k(1+\delta_k^2)\right)\right] \sim \alpha - y + \sum_{k=1}^{m}\gamma_k(1+\delta_k^2),$$

in other words,

$$\lim_{w\to 0} f(w,y) = \alpha - y + \sum_{k=1}^{m}\gamma_k(1+\delta_k^2), \tag{12}$$

so the right-hand side of (12) may be thought of as $f(0, y)$.

Next, fix some y and denote I the integral in (6). Besides that, denote $I(u)$ the exact value of the integral over $[0, u]$:

$$I(u) = \int_0^u f(w,y)\,dw, \tag{13}$$

and let $\Delta_1(u)$ be the error of truncation

$$\Delta_1(u) = |I - I(u)|. \tag{14}$$

Clearly the error is decreasing function of u, and $\Delta_1(u) \to 0$ as $u \to \infty$. Denote $\overline{\Delta}_1(u)$ an upper bound for $\Delta_1(u)$: $\Delta_1(u) \le \overline{\Delta}_1(u)$, $u \in [0, \infty)$, which has a convenient analytical representation, and behaves as Δ_1, that is, decreases in u, and $\overline{\Delta}_1(u) \to 0$ as $u \to \infty$.

Next, denote $I_n(u)$ the result of numeric integration (with partition of $[0, u]$ into n segments) of the integral $I(u)$, let $\Delta_2(u,n) = |I(u) - I_n(u)|$ be the error of numeric integration, and $\overline{\Delta}_2(u,n)$ be an upper bound for the latter with the following properties: $\overline{\Delta}_2$ increases in u, decreases in n, and

$$\lim_{u\to 0}\overline{\Delta}_2(u,n) = 0,\quad \lim_{u\to\infty}\overline{\Delta}_2(u,n) = \infty,\quad \lim_{n\to\infty}\overline{\Delta}_2(u,n) = 0.$$

Usually upper bounds for error of numeric integration have the form

$$\overline{\Delta}_2(u,n) = \frac{g(u)}{n^p}, \tag{15}$$

where g increases, and $p > 0$. The following theorem is true for such bounds.

Theorem 1 *Let the truncation error $\overline{\Delta}_1(u)$ be a decreasing function and tend to 0 as $u \to \infty$. Let the error of numeric integration have the form (15), where the function g is strictly increasing, takes positive values for $u > 0$, and tends to ∞ as $u \to \infty$, and $p > 0$. Denote u_n a solution of the optimization problem*

$$\overline{\Delta}_1(u) + \overline{\Delta}_2(u,n) \longrightarrow \min_u$$

under fixed n. Then the sequence u_n is nondecreasing, $u_n \to \infty$ and $\overline{\Delta}_1(u_n)+\overline{\Delta}_2(u_n,n) \to 0$ as $n \to \infty$.

Now consider calculating bounds $\overline{\Delta}_1$ and $\overline{\Delta}_2$. There are many error bounds for numeric integration methods, that hold under various conditions. For example, if a function f is k times differentiable (with some integer $k > 0$) and its k-th derivative is bounded:

$$\sup_{x\in[0,u]} |f^{(k)}(x)| \le L_k,$$

then the trapezoidal method for calculation the integral

$$\int_0^u f(w)\,dw$$

leads to the following error bounds (see [3]) corresponding to values $k = 1$ and 2 respectively:

$$\overline{\Delta}_2^{(1)}(u,n) = \frac{L_1 u^2}{4n}, \quad \overline{\Delta}_2^{(2)}(u,n) = \frac{L_2 u^3}{12n^2}\,.$$

Both bounds are in the class (15).

The following auxiliary results are useful for calculation of the truncation error (14).

Lemma 1 *Let $b(w)$ be an integrable upper bound for the integrand f, that is, $|f(w)| \le b(w)$, $w \ge 0$. Then an upper bound for truncation error has the form*

$$\overline{\Delta}_1(u) = \int_u^\infty b(w)\,dw.$$

Lemma 2 *Let $g(w)h(w)$ be an integrable upper bound for the integrand f, that is, $|f(w)| \le g(w)h(w)$, $w \ge 0$, where g is non-increasing, h is integrable, and*

$$H(u) = \int_u^\infty h(w)\,dw.$$

Then

$$\overline{\Delta}_1(u) = g(u)H(u)$$

is an upper bound for truncation error.

Obtaining specific bounds for errors of truncation and numeric integration requires cumbersome calculations of bounds for derivatives. However, it is conceptually simple, and will not be presented here.

Note two strategies of optimization of approximate calculation of the integral (6). First starts with fixed number of nodes n in numeric integration lattice (this essentially represents computational expenses), and calculates the best truncation point u_n, which was defined in theorem 1. Another strategy starts with the required precision ε, and looks for the minimal value of n, providing the precision.

References

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