Characteristic Classes of Families of Risk Measures

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Abstract

Decision-making under risk is usually implemented using a functional (risk measure) defined on a set of probability distributions and representing preference relation on the set. It is desirable to solve an inverse problem – constructing a risk measure representing given preference relation – on as narrow set of distributions as possible. The present paper introduces a concept of characteristic class of distributions for a family of risk measures, possessing the property that continuation of a risk measure from characteristic class to the whole set of distributions is unique within the family. Characteristic classes have been completely described for families of expected utility measures, distorted probability measures and combined functionals.

1 Introduction

Decision-making under uncertainty and risk involves studying an (individual) preference relation over probability distributions; the best solution corresponds to the best (in the sense of preference) distribution among those attainable. In many cases preference relation may be represented by a real-valued functional defined on the set of distributions.

A model of individual preference (or representing functional) appears as a result of solving an inverse problem: constructing a functional via observations of decision-making process. Observations may be implemented in passive or active mode; in the latter case it is desirable to reduce cost of experiments by their rational organization.

The concept of characteristic class of distributions for a given family of functionals (risk measures) relates to the inverse problem. A characteristic class is a class of distributions such that defining a functional on that class completely defines the whole functional from the given family. In other words, continuation of a functional from characteristic class to the whole set of distributions is unique within the family of functionals. Thus solving an inverse problem reduces to the characteristic class.

The current paper provides a strict definition of characteristic class and presents calculation of characteristic classes for families of expected utility functionals, distorted probability functionals and combined functionals.

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2 Basic concepts

Let $\mathcal{F}$ be a set of distribution function on the set of reals $\mathbf{R}$. In the present paper we will assume that $\mathcal{F}$ contains all distribution function with bounded support[2]. Let $\preceq$ be a preference relation on $\mathcal{F}$, that is, a complete transitive binary relation [1]. In decision-making problems this relation is usually treated as follows: for $F, G \in \mathcal{F}$ relation $F \preceq G$ means that the distribution F is not better (from the decision-maker's point of view) than the distribution G.

Under some regularity conditions the preference relation may be represented by a functional (risk measure) $\mu : \mathcal{F} \to \mathbf{R}$ [1] in the sense

$$F \preceq G \iff \mu(F) \leq \mu(G), \quad F, G \in \mathcal{F}. \tag{1}$$

Knowing the functional μ allows calculating optimal decisions. However, precise calculation of the functional μ via preference relation $\preceq$ through (1) is a complex inverse problem. The problem may be simplified by restricting it to a subset of the set of distributions $\mathcal{F}$. If no information is available about μ, then such restriction is impossible. Sometimes additional information is available in a form of a class of preference relations, described by a set of axioms, to which the relation $\preceq$ belongs. The information makes simplifying the inverse problem possible.

In the present paper the concept of characteristic class for a family of risk measures is being introduced. Continuation of a risk measure from the characteristic class to the whole set of distributions $\mathcal{F}$ is unique within the given family of risk measures.

Denote $\mathcal{M}$ the set of all real-valued functionals (risk measures) on $\mathcal{F}$.

For $a \in \mathbf{R}$ denote

$$W_a(x) = \begin{cases} 0, & x < a, \\ 1, & a \leq x \end{cases}$$

a degenerate (at a point $a \in \mathbf{R}$) distribution function. Let

$$\mathcal{W} = \{W_a,\, a \in \mathbf{R}\} \tag{2}$$

be the set of all degenerate distributions.

For $a < b$ and $p \in (0, 1)$ denote

$$B_{a,b,p}(x) = \begin{cases} 0, & x < a, \\ 1 - p, & a \leq x < b, \\ 1, & b \leq x \end{cases}$$

a Bernoulli distribution function. Next, denote

$$\mathcal{B}_{a,b} = \{B_{a,b,p},\ p \in (0, 1)\} \tag{3}$$

[2] A distribution function F on $\mathbf{R}$ has bounded support $[a, b]$, if $-\infty < a < b < \infty$, $a = \sup\{x : F(x) = 0\}$, $b = \inf\{x : F(x) = 1\}$.

the set of all such distributions with a, b fixed. In particular, if $a = 0, b = 1$ then $\mathcal{B} = \mathcal{B}_{0,1}$.

Let $U : \mathbf{R} \to \mathbf{R}$ be a real function. The functional $\rho_U : \mathcal{F} \to \mathbf{R}$, defined by

$$\rho_U(F) = \int_{-\infty}^{\infty} U(x)\, dF(x), \quad F \in \mathcal{F}, \tag{4}$$

is called an expected utility functional, and U is called an utility function. Denote $\mathbf{U}$ the set of all increasing utility functions U, and $\mathcal{U} = \{\rho_U,\ U \in \mathbf{U}\}$ the corresponding family of expected utility functionals. Note that functionals in $\mathcal{U}$ are strictly monotone with respect to the first stochastic dominance [2].

If X_F is a random variable with distribution function F, then $\rho_U(F) = \mathbf{E}U(X_F)$. Expected utility functional may be also written as

$$\rho_U(F) = \int_0^1 U(F^{-1}(v))\, dv, \quad F \in \mathcal{F},$$

where

$$F^{-1}(v) = \sup\{x : F(x) \le v\} \tag{5}$$

is the inverse function for $F \in \mathcal{F}$. In particular, for degenerate and Bernoulli distributions one has

$$W_a^{-1}(v) = a, \quad v \in (0,1); \quad B_{a,b,p}^{-1}(v) = \begin{cases} a, & v \in (0, 1-p), \\ b, & v \in [1-p, 1) \end{cases} \tag{6}$$

Let $g : [0,1] \to [0,1]$ be a nondecreasing function with

$$g(0) = 0, \quad g(1) = 1, \tag{7}$$

and $\mathbf{G}$ be the set of all such functions. A functional $\pi_g : \mathcal{F} \to \mathbf{R}$, defined by

$$\pi_g(F) = \int_{-\infty}^0 [g(1 - F(x)) - 1]\, dx + \int_0^{\infty} g(1 - F(x))\, dx \tag{8}$$

is called a distorted probability functional (risk measure). It may also be written as

$$\pi_g(F) = -\int_0^1 F^{-1}(v)\, dg(1-v) = \int_0^1 F^{-1}(v)\, d\tilde{g}(v),$$

where $\tilde{g}(v) = 1 - g(1-v)$, $v \in [0,1]$ is a dual distortion function for g. Denote $\mathcal{G} = \{\pi_g : g \in \mathbf{G}\}$ the family of all distorted probability functionals.

In [3] a combined risk measure

$$\mu_{U,g}(F) = \int_0^1 U(F^{-1}(v))\, d\tilde{g}(v), \quad F \in \mathcal{F}, \tag{9}$$

was introduced. It inherits attractive features of expected utility and distorted probability functionals. The functional (9) possesses more flexibility as compared with basic risk measures. The functional allows representing nonlinearity of preferences both in the space of distributions and in the space of random variables. The properties endow the functionals (9) the ability to adequately describe individual risk aversion [4].

3 Characteristic classes

Let the risk measure $\mu \in \mathcal{M}$, representing the preference relation $\preceq$ on $\mathcal{F}$, be a member of a family of functionals $\mathcal{N} \subseteq \mathcal{M}$. We will call the class of distributions $\mathcal{F}(\mathcal{N}) \subseteq \mathcal{F}$ a characteristic class for the family $\mathcal{N}$, if $\mu(F) = \nu(F)$, $F \in \mathcal{F}(\mathcal{N})$ and $\mu, \nu \in \mathcal{N}$ imply $\mu = \nu$.

Let us derive a few properties of a characteristic class. First, it is clear that any family of risk measures $\mathcal{N}$ possesses a characteristic class $\mathcal{F}$, so the concept is not empty. Of course, "minimal" (in some sense) classes are most valuable. Next, if a family of risk measures consists of a single element μ, then the empty class may be chosen as characteristic. The concept of characteristic class becomes meaningful when $\mathcal{N}$ contains more that one element, and does not coincide with $\mathcal{M}$.

Proposition 1 *Let $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \mathcal{M}$. Then there are characteristic classes $\mathcal{F}(\mathcal{N}_1), \mathcal{F}(\mathcal{N}_2)$ such that $\mathcal{F}(\mathcal{N}_1) \subseteq \mathcal{F}(\mathcal{N}_2)$.*

Proof. Let $\mathcal{F}(\mathcal{N}_2)$ be a characteristic class for the family $\mathcal{N}_2$, and $\mathcal{N}_1 \subseteq \mathcal{N}_2$. Then $\mathcal{F}(\mathcal{N}_1)$ may be chosen to be equal to $\mathcal{F}(\mathcal{N}_2)$.

Remark 1 *Note that a characteristic class of a family is not unique, so it is possible under conditions of the proposition 1 that $\mathcal{F}(\mathcal{N}_1) \not\subseteq \mathcal{F}(\mathcal{N}_2)$ for $\mathcal{N}_1 \subseteq \mathcal{N}_2$.*

Now let us calculate characteristic classes for families of expected utilities (4) and distorted probabilities (8).

Theorem 1 *The class of degenerate distributions $\mathcal{W}$ defined in (2) is a characteristic class for the expected utilities family.*

Proof. Indeed, calculating the functional (4) on a degenerate distribution $W_a \in \mathcal{W}$, one gets

$$\rho_U(W_a) = \int_{-\infty}^{\infty} U(x)\, dW_a(x) = U(a),$$

so the values $\rho_U(W_a)$ for all $a \in \mathbf{R}$ completely define the function U, hence also the functional ρ_U.

Remark 2 *Note that since each U is monotone, it is sufficient to use the class $\mathcal{W}^{\mathbf{Q}}$ of degenerate distributions W_a with rational a. In this case the characteristic class is countable.*

Theorem 2 *The class of Bernoulli distributions $\mathcal{B} = \mathcal{B}_{0,1}$, defined in (3), is a characteristic class for distorted probabilities family (8).*

Proof. Indeed, calculating π_g for $B_{0,1,p} \in \mathcal{B}$, one gets from (6):

$$\pi_g(B_{0,1,p}) = 0 \cdot \tilde{g}(1-p) + 1 \cdot (1 - \tilde{g}(1-p)) = g(p), \quad p \in (0,1),$$

which together with (7) completely defines the function g, hence the functional π_g.

Remark 3 *The function g is monotone, so it is sufficient to define it in rational points of $(0,1)$, thus the characteristic class may be chosen as the set $\mathcal{B}^{\mathbf{Q}}$ of Bernoulli distributions B_p with rational values $p \in (0,1)$. In this case the characteristic class, as in remark 2, is countable.*

Theorem 3 *Let $\mathcal{M}$ be the family of combined risk measures (9), where U is a strictly increasing utility function. Let the combined functional $\mu_{U,g}$ be strictly monotone with respect to the first stochastic dominance. Then $\mathcal{W} \cup \mathcal{B}$ is a characteristic class for $\mathcal{M}$.*

Proof. Calculate $\mu_{U,g}$ on distributions $\mathcal{W}$ and $\mathcal{B}$:

$$\mu_{U,g}(W_a) = \int_0^1 U(a)\, d\tilde{g}(v) = U(a), \quad a \in \mathbf{R}$$

and

$$\mu_{U,g}(B_p) = U(0)\tilde{g}(1-p) + U(1)(1-\tilde{g}(1-p)) = U(0) + g(p)(U(1) - U(0)).$$

Now knowing all values $\mu_{U,g}(W_a)$, $a \in \mathbf{R}$ and $\mu_{U,g}(B_p)$, $p \in (0,1)$ allows sequential calculating

$$U(a) = \mu_{U,g}(W_a),\ a \in \mathbf{R}$$

and

$$g(p) = \frac{\mu_{U,g}(B_p) - \mu_{U,g}(W_0)}{\mu_{U,g}(W_1) - \mu_{U,g}(W_0)}, \quad p \in (0,1),$$

which completely define the functional parameters U, g, hence also the functional $\mu_{U,g}$.

Remark 4 *As in partial cases presented in theorems 1 and 2 (see remarks 2 and 3), the last theorem remains valid with the countable subclass $\mathcal{W}^{\mathbf{Q}} \cup \mathcal{B}^{\mathbf{Q}}$.*

Conclusion. A concept of characteristic class for a family of risk measures has been introduced and studied in the paper. The concept allows simpler solving inverse problems of risk theory. Characteristic classes have been calculated for classic families of expected utilities and distorted probabilities, and for a new family of combined risk measures.

Describing characteristic classes for families of preference relations is the next interesting step in the direction.

References

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