CANONICAL REPRESENTATION OF PREFERENCES OVER A SET OF PROBABILITY DISTRIBUTIONS

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Abstract

Building a model of individual preferences is a key for rational decision-making under uncertainty. Since preference can hardly be studied completely, its approximation from partially known preference is very important. The present paper provides a framework for building such approximation for regular preferences on abstract partially ordered set, and then applies the results to preferences on the set of distributions, thus establishing a link to decision-making.

Keywords

Decision-making, risk, individual preferences, partial preference, risk measure, canonical representation, expected utility, distorted probability, combined functional

Introduction

Decision-making under risk usually reduces to choosing the best probability distribution in a predefined set of distributions [1], where "the best" means the most preferable distribution in the sense of individual preferences. The preferences may often be represented by a real functional [1, 2] called risk measure; a classic representation of the sort is the representation of linear preferences by expected utility functional [3]. As a result, the most preferable distribution may be found by solving a restricted optimization problem.

In practice complete preference relation can hardly be discovered, especially in nonlinear (non-expected-utility) case. Only some partial preference is usually observable. Thus the problem of approximation of the complete preference by a partial one arises. Moreover, the approximation should allow concordant representation of preferences by risk measures.

The present paper is devoted to constructing a framework for concordant continuation of partial preference and, on the other hand, for restriction of a preference to a partial preference in such a way that families of representing risk measures are also concordant in some sense. The problem is studied in an abstract set first, and then the results are applied to preference relations over probability distributions.

Preferences and canonical representing functionals

Let $\mathbf{X}$ be a set. Recall that a binary relation R on $\mathbf{X}$ is any subset in the Cartesian product $R \subseteq \mathbf{X} \times \mathbf{X}$. We say that elements $x, y \in \mathbf{X}$ are in relation R, if $(x, y) \in R$, and sometimes write this fact as xRy. A relation R is called *reflexive*, if it contains the diagonal $I_X = \{(x,x), x \in X\}$ of $\mathbf{X} \times \mathbf{X}$. The relation $R^T = \{(x,y) : (y,x) \in R\}$ is called a *transposition* of R, $R^s = R \cap R^T$ is called a *symmetric* part of R, and $R^a = R \setminus R^s$ is called an *asymmetric* part of R. A relation R is called *complete* if for any pair $(x, y) \in \mathbf{X} \times \mathbf{X}$ at least one of pairs (x, y) and (y, x) belongs to R, that is $R \cup R^T = \mathbf{X} \times \mathbf{X}$. A relation R is called transitive if $(x, y) \in R$, $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in \mathbf{X}$. A complete transitive relation on $\mathbf{X}$ is called *preference* and is denoted by $\prec=$. Its symmetric part is reflexive, transitive and symmetric, in other words, it is an *equivalence* relation, which is denoted by $\sim$. Asymmetric part of a preference relation is a strict preference, and is denoted by $\prec$. A reflexive transitive binary relation on $\mathbf{X}$ is called *partial*

preference. Note that any complete relation is also reflexive, so preference is a special case of partial preference.

A functional $\mu : \mathbf{X} \to R$ represents the preference relation $\prec=$ on $\mathbf{X}$ if

$$\mu(x) \le \mu(y) \Leftrightarrow x \prec= y; \quad x, y \in \mathbf{X}. \tag{1}$$

Note that if f is any strictly increasing real function, then functionals μ and ν related by

$$\nu(x) = f(\mu(x)), x \in \mathbf{X}, \tag{2}$$

represent the same preference relation. Thus representing functional is not uniquely defined, and there is a need in defining *canonical* representing functional in such a way, that it is uniquely defined for any appropriate preference relation. Such a functional is introduced below using structure of partially ordered set. Note, that if the two functionals μ and ν represent the same preference relation, they are necessarily related by (2) with some strictly increasing function f. Note also that a partial preference $\prec=$ may be represented by real functionals as in (1). However representation is essentially not unique in this case: no relation of the sort (2) is required for different functionals representing the same partial preference.

Let $(\mathbf{X}, \le)$ be a partially ordered set. Recall that *chain* in this set is any linearly ordered subset. The partial preference relation $\prec=$ on $\mathbf{X}$ is called *concordant* with the partial order $\le$, if for any $x, y \in \mathbf{X}$ the implication $x < y \Rightarrow x \prec y$ is valid. In what follows we will consider only concordant preferences on $(\mathbf{X}, \le)$.

Definition 1. Let $(\mathbf{X}, \le)$ be a partially ordered set, $A \subseteq \mathbf{X}$ be a fixed chain in $(\mathbf{X}, \le)$. A preference relation $\prec=$ on $\mathbf{X}$ is called regular, if it is concordant with the partial order $\le$ and equivalence classes (with respect to $\sim$, the symmetric part of $\prec=$) possess nonempty intersections with A:

$$C \cap A \ne \varnothing, C \in \mathbf{X} / \sim. \tag{3}$$

In what follows we assume all preference relations to be regular with respect to a fixed chain A.

Theorem 1. Let $(\mathbf{X}, \le)$ be a partially ordered set, $A \subseteq \mathbf{X}$ be a fixed chain in $(\mathbf{X}, \le)$, and $f : A \to \mathbf{R}$ be a strictly increasing function. Then any regular preference relation $\prec=$ on $\mathbf{X}$ is uniquely represented by a functional $\mu = \mu_{\prec=}$ such that

$$\mu(x) = f(x), x \in A. \tag{4}$$

Note that Theorem 1 implicitly states existence of representing functional. Condition (4) assures uniqueness of representing functional, which may be called *canonical*. This condition makes it clear why there are many representing functionals for a single preference relation: different functionals may be obtained by selecting different arbitrary functions f in (4).

Now let's introduce concordant partial preferences.

Definition 2. Let P and Q be two partial preferences on $(\mathbf{X}, \le)$, concordant with the order $\le$. P is called a *subpreference* of Q, if $P^s \subseteq Q^s$ and $P^a \subseteq Q^a$.

For a regular preference R denote $\mathbf{P}_R$ the set of all subpreferences of R. It can be easily seen that for any partial preference $Q \in \mathbf{P}_R$ one has $(\le) \subseteq Q \subseteq R$, so R is the maximal element of $\mathbf{P}_R$, and $I_{\mathbf{X}}$ is the minimal element of $\mathbf{P}_R$ with respect to inclusion. Moreover, $(\le)^s = I_{\mathbf{X}} \subseteq Q^s \subseteq R^s$ and $(\le)^a = (<) \subseteq Q^a \subseteq R^a$. Theorem 1 states existence of the canonical

representing functional μ_R. Now consider representation of partial preferences from $\mathbf{P}_R$ by canonical functionals. For a $Q \in \mathbf{P}_R$ denote $\mathbf{M}(Q)$ the collection of all functionals $\mu_Q : \mathbf{X} \to \mathbf{R}$ such that

$$\mu_Q(x) = f(x), x \in A. \tag{5}$$

Theorem 2. Let $(\mathbf{X}, \leq)$ be a partially ordered set, $A \subseteq \mathbf{X}$ be a fixed chain in $(\mathbf{X}, \leq)$, and $f : A \to \mathbf{R}$ be a strictly increasing function. Let R be a regular (with respect to A) preference relation on $\mathbf{X}$. Then for each $Q \in \mathbf{P}_R$ the set of canonical representing functionals $\mathbf{M}(Q)$ is non-empty. Moreover, if $P, Q \in \mathbf{P}_R$ and P is a subpreference of Q, then $\mu_R \in \mathbf{M}(Q) \subseteq \mathbf{M}(P)$.

This theorem describes the "dynamics" of sets of representing functionals: the more one knows about preference, the narrower is the set of representing functionals. In particular, this allows building approximate solutions for decision-making problems in presence of incomplete information about preference.

Now consider the inverse question: given some partial preference Q, what is the set $\mathbf{P}^Q$ of true preference relations, which have Q as their subpreference? The answer is provided by the following theorem.

Theorem 3. Let $(\mathbf{X}, \leq)$ be a partially ordered set, $A \subseteq \mathbf{X}$ be a fixed chain in $(\mathbf{X}, \leq)$, and $f : A \to \mathbf{R}$ be a strictly increasing function. Let R be a regular preference relation on $\mathbf{X}$ and let Q be its subpreference. Then $R \in \mathbf{P}^Q$ and $\mathbf{P}^Q$ coincides with the set of all preference relations R_μ, which may be represented by functionals $\mu \in \mathbf{M}(Q)$ by the rule

$$(x, y) \in R_\mu \Leftrightarrow \mu(x) \leq \mu(y); x, y \in \mathbf{X}. \tag{6}$$

Each preference relation in $\mathbf{P}^Q$ is regular with respect to A.

Preference on a set of distributions

Now let us apply the just built abstract construction to decision-making framework. Let $\mathbf{F}$ be a set of real distribution functions with bounded support, and let $\prec=$ be a preference relation on $\mathbf{F}$. Decision-making under risk means choosing the best distribution F^* in the set $\hat{\mathbf{F}}$ of attainable distributions, that is, $F \prec= F^*$ for all $F \in \hat{\mathbf{F}}$. The problem reduces to a standard optimization problem by introducing a representing functional $\mu : \mathbf{F} \to \mathbf{R}$ for $\prec=$ in the sense

$$\mu(F) \leq \mu(G) \Leftrightarrow F \prec= G; F, G \in \mathbf{F}. \tag{7}$$

In practice the preference relation is usually known only partially, so there is a problem of approximation of μ using partial preference.

For $F, G \in \mathbf{F}$ we say that G stochastically dominates F if $F(x) \geq G(x)$ for all $x \in \mathbf{R}$. This fact is denoted by $F \leq_1 G$. It is well known [4], that stochastic dominance is a partial order on a set of probability distributions. For each $a \in \mathbf{R}$ we define the degenerate distribution function by

$$W_a(x) = \begin{cases} 0, & x < a, \\ 1 & x \geq a. \end{cases}$$

It can be easily seen that the set of all degenerate distribution functions $\mathbf{W} = \{W_a, a \in \mathbf{R}\}$ form a chain in the partially ordered set $(\mathbf{F}, \leq_1)$.

Now let $\prec=$ be a preference relation on $(\mathbf{F},\leq_1)$ which is regular with respect to $\mathbf{W}$. In this case each equivalence class (with respect to symmetric part of the preference) contains exactly one degenerate distribution function W_a; the value of parameter a of this function is called *certainty equivalent* of any distribution from the equivalence class. Let us define the function f in (4) to be equal to the certainty equivalent

$$f(W_a) = a, a \in \mathbf{R}. \tag{8}$$

Theorem 4. Let a preference relation $\prec=$ be defined on the set of probability distributions $(\mathbf{F},\leq_1)$, and let it be regular (with respect to $\mathbf{W}$). Then there exists the functional $\mu:\mathbf{F}\to\mathbf{R}$ representing preference relation $\prec=$ and satisfying canonical condition (8).

In applications human preferences are discovered via polls or observations of actual decision-making under risk. This would lead to establishing proper relations of the sort $F \prec G$ or $F \sim G$ for a (finite) number of pairs of distributions (F,G). Thus only partial preference is available< though it is concordant with the "true" preference. In this case the class of functionals represents the preference.

Theorem 5. Let a preference relation $\prec=$ be defined on the set of probability distributions $(\mathbf{F},\leq_1)$, and let it be regular (with respect to $\mathbf{W}$). Further, let partial preference $\prec=^*$ be concordant with $\prec=$. Then the class $M(\prec=^*)$ of functionals, representing $\prec=^*$ and satisfying the condition (8), is not empty, and contains the functional, which represents $\prec=$.

Theorem 6. Let a preference relation R be defined on the set of probability distributions $(\mathbf{F},\leq_1)$, and let it be regular (with respect to $\mathbf{W}$). Further, let a partial preference Q be concordant with R. Then $R \in \mathbf{P}^Q$, where $\mathbf{P}^Q$ denotes the collection of all preference relations R_μ, which are induced by functionals $\mu \in \mathbf{M}(Q)$ by the rule

$$(F,G) \in R_\mu \Leftrightarrow \mu(F) \leq \mu(G).$$

Consider a few examples of regular preference relations. In [3] it was shown, that if the preference relation $\prec=$ on $(\mathbf{F},\leq_1)$ satisfies some linearity axioms, then it may be represented by an expected utility functional

$$\rho(F) = \int U(x)dF(x), F \in \mathbf{F},$$

where U stands for utility function. Regularity of this linear preference is implied by concordance with first stochastic dominance; utility function U turns out strictly increasing, and canonical functional has the form

$$\mu(F) = U^{-1}(\rho(F)), F \in \mathbf{F}.$$

Another example is given by preferences induced by coherent risk measures, in particular, distorted probability functionals [5]. The latter has the form

$$\pi(F) = -\int_0^1 F^{-1}(v)dh(1-v), F \in \mathbf{F},$$

where $h:[0,1]\to[0,1]$ is strictly increasing distortion with boundary values $h(0)=0$, $h(1)=1$. Similar examples may be built within class of combined risk measures [6].

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