ПРЕДСТАВЛЕНИЕ ПРЕДПОЧТЕНИЙ НА МНОЖЕСТВЕ РИСКОВ СЕМЕЙСТВАМИ ФУНКЦИОНАЛОВ
REPRESENTATION OF PREFERENCES ON A SET OF RISKS BY FUNCTIONAL FAMILIES

Arcady Novosyolov

Institute of computational modeling SB RAS
ICM SB RAS, Academgorodok, Krasnoyarsk, Russia, 660036
anov@icm.krasn.ru

Quantitative decision-making under risk often reduces to solving optimization problems. The reduction requires representation of decision-maker's preferences by real-valued functionals. Personal preferences over risky outcomes in practice are seldom known completely. Partial preference is a more reasonable object to be assumed known. The paper is devoted to representation of personal preferences over sets of probability distributions by real functionals and functional families, and to approximation of complete preferences by partial ones in terms of representing families of functionals.

Keywords: risk, distribution, decision-making, preference, risk measure, concordance, representation, stochastic dominance, approximation, expected utility, coherent risk measure, partition, factor-set, canonical representation.

Representation of personal preferences over risks by real valued functionals is a key and challenging problem in risk theory [1]. It allows converting a qualitative decision-making problem into a quantitative optimization problem. The problem was first tackled in the seminal book [2] where a representation theorem was established for linear preferences. Nonlinear case has brought much less attention until recently. A few variants of general representation theorems were presented in [1, 3]. Some related results for preferences described in terms of admissible sets of risks may be found in [4] and later papers on coherent risk measures and similar functionals [5,6].

Nonlinearity of preferences is not the only trouble in representation. Complete preference usually cannot be extracted as a whole. What one knows is at most partial preference. This brings more problems to be solved. How can one represent a partial preference? How this should be done in a consistent way so that partial representation would serve approximation for the complete representation?

The present paper addresses these questions in detail. First a representation theorem for a complete regular preference is established. Then a similar result is obtained for partial preferences, and finally a procedure of completing a partial preference is described; the latter establishes concordance between complete and partial preferences, and turns partial representation into an approximation tool. All representations are formulated in terms of canonical functionals; this makes the approximation direct, without need in additional tuning of algorithms.

The paper is organized as follows. Section 1 provides some basic concepts on relations and probability distributions, including factor-sets and mutual representation of relations and structured partitions of the set of interest. Section 2 contains introduction to preference relations and related structures. Section 3 describes regular preferences. Section 4 is devoted to representation results for complete and partial preferences. Illustrating examples are presented in section 5 .

1. Preliminaries

Let $\mathbf{X}$ be any set. A partition T of $\mathbf{X}$ is a family of non-empty disjoint in pairs sets $T = \{K_\lambda, \lambda \in \Lambda\}$ such that $\sum_{\lambda\in\Lambda} K_\lambda = \mathbf{X}$. A partition T is called *slicing* of a partition S if for each $A \in T$ there exists $B \in S$ such that $A \subseteq B$. S is called *enlargement* of T.

A binary relation Q on $\mathbf{X}$ is any subset in the cartesian product $Q \subseteq \mathbf{X}\times\mathbf{X}$. Given a relation Q, its transpose is defined by $Q^T = \{(x,y):(y,x)\in Q\}$, and *symmetric* and *asymmetric* parts by $Q^s = Q \cap Q^T$ and $Q^a = Q \setminus Q^s$ respectively. For two relations Q, R their *composition* is defined by $Q\circ R = \{(x,y): \exists z \in \mathbf{X} \mid (x,z)\in Q, (z,y)\in R\}$. R is called *continuation* of Q (or Q is called *restriction* of R) if $Q^s \subseteq R^s$ and $Q^a \subseteq R^a$. The subset $I_\mathbf{X} = \{(x,x), x\in\mathbf{X}\}$ is called the *diagonal*, it represents *equality* relation on $\mathbf{X}$. A relation Q is called *reflexive* if $I_\mathbf{X} \subseteq Q$, *symmetric* if $Q = Q^T$, *transitive* if $Q\circ Q \subseteq Q$, *antisymmetric* if $Q\cap Q^T \subseteq I_\mathbf{X}$, *complete* if $Q \cup Q^T = \mathbf{X}\times\mathbf{X}$.

Reflexive, symmetric and transitive relation Q on $\mathbf{X}$ is called *equivalence*. An inclusion $(x,y)\in Q$ is also denoted by $x \sim y$ in this case. Any equivalence generates a partition of $\mathbf{X}$ to disjoint equivalence classes $\mathbf{X} = \sum_{\lambda\in\Lambda} K_\lambda$. Here $x, y \in K_\lambda$ (for some $\lambda \in \Lambda$) iff $x \sim y$. The set of all equivalence classes $\widetilde{\mathbf{X}}_Q = \{K_\lambda, \lambda\in\Lambda\}$ is called a *factor-set*. Let Q, R be two equivalence relations. If R is a continuation of Q then $\widetilde{\mathbf{X}}_Q$ is a slicing of $\widetilde{\mathbf{X}}_R$.

Reflexive, antisymmetric and transitive relation Q on $\mathbf{X}$ is called *(partial) order* and is denoted by $\leq$. Symmetric part of an order coincides with $I_\mathbf{X}$, and asymmetric part represents strict order $<$. A complete order is also called *linear*.

Let $\mathbf{R}$ be the set of reals and $\mathbf{B}$ - the σ-algebra of its Borel subsets. A *probability distribution* on a measurable space $(\mathbf{R},\mathbf{B})$ is any non-negative σ-additive set function P on $\mathbf{B}$ with $P(\mathbf{R}) = 1$. A probability distribution is called *bounded* if there exist such $-\infty < a < b < \infty$ that $P([a,b]) = 1$. Denote the set of all bounded probability distributions by $\mathbf{P}$. *Lower* $l(P)$ and *upper* $u(P)$ bounds of a probability distribution $P \in \mathbf{P}$ are defined by $l(P) = \sup\{a\in\mathbf{R}: P((-\infty,a]) = 0\}$, $u(P) = \inf\{b\in\mathbf{R}: P([b,\infty)) = 0\}$. Segment $[l(P), u(P)]$ is called the support of P.

Each distribution $P \in \mathbf{P}$ may be described by different tools. One of them is *distribution function* $F = F_P$ which is defined by $F_P(x) = P((-\infty, x])$, $x \in \mathbf{R}$. There is a one-to-one correspondence between distributions and distribution functions. Therefore in what follows we will treat $\mathbf{P}$ as the set of distribution functions F with lower and upper bounds

$$l(F) = \sup\{a\in\mathbf{R}: F(a) = 0\}, \quad u(F) = \inf\{b\in\mathbf{R}: F(b) = 1\}.$$

The *inverse* distribution function F^{-1} of F is defined by $F^{-1}(v) = \sup\{x : F(x) \leq v\}$ for $v \in [0,1)$ and $F^{-1}(1) = u(F)$. *Mean* or *expectation* $\mathbf{E}F$ of a distribution function F is

$$\mathbf{E}F = \int_{-\infty}^{\infty} x dF(x) = \int_{l(F)}^{u(F)} x dF(x) = \int_0^1 F^{-1}(v) dv.$$

Degenerate distribution function W_a with a jump at point a is defined by $W_a(x)=0$ for $x<a$ and $W_a(x)=1$ for $x \ge a$. Denote $\mathbf{W}=\{W_a, a\in\mathbf{R}\}$ the set of all degenerate distributions. Clearly $\mathbf{W}\subseteq\mathbf{P}$, $l(W_a)=u(W_a)=\mathbf{E}W_a=a$, $a\in\mathbf{R}$ and for any non-degenerate distribution function $F\in\mathbf{P}$ we have $l(F)<u(F)$.

2. Preferences

Complete transitive relation Q on $\mathbf{X}$ is called *preference*. A relation $(x,y)\in Q$ we also denote by $x \prec= y$. Symmetric part Q^s of a preference Q is equivalence relation, and its asymmetric part Q^a represents strict preference $\prec$. Factor-set $\widetilde{\mathbf{X}}_Q$, generated by the symmetric part of Q is supplied with linear order: $X<Y$ if $x\prec y$ for some (hence any) $x\in X, y\in Y$. So we see that preference relation on $\mathbf{X}$ generates linear ordered partition of $\mathbf{X}$. It turns out that converse is also true, so we have a tool for constructing preference relations which is described in the following lemma.

Lemma 1. Let $\widetilde{\mathbf{X}}=\{K_\lambda, \lambda\in\Lambda\}$ be a linearly ordered partition of $\mathbf{X}$. Then there exists the preference relation Q on $\mathbf{X}$ such that it produces $\widetilde{\mathbf{X}}$ as the factor-set: $\widetilde{\mathbf{X}}=\widetilde{\mathbf{X}}_Q$.

Another tool for generating a preference relation on $\mathbf{X}$ consists in defining a functional $f:\mathbf{X}\to\mathbf{R}$, it is described in the following lemma.

Lemma 2. A relation $\prec=$ defined by a functional $f:\mathbf{X}\to\mathbf{R}$ via

$$x \prec= y \Leftrightarrow f(x)\le f(y)$$

constitutes a preference relation Q_f on $\mathbf{X}$.

Conversion of lemma 2 would provide a representation theorem for preference relation. It is possible only with additional regularity assumptions, and will be implemented in section 4. Denote Q_f^s and Q_f^a symmetric and asymmetric parts of the preference relation generated by f:

$$Q_f^s=\{(x,y)\in\mathbf{X}\times\mathbf{X}: f(x)=f(y)\},\ Q_f^a=\{(x,y)\in\mathbf{X}\times\mathbf{X}: f(x)<f(y)\}. \qquad (1)$$

A reflexive transitive relation Q on $\mathbf{X}$ is called *partial preference*. Complete preference and partial order are its partial cases. Symmetric part Q^s of a partial preference Q is equivalence relation, so it generates the factor-set $\widetilde{\mathbf{X}}_Q$. The latter is partially ordered in this case. Converse statement is true, we formulate it in the following lemma.

Lemma 3. Let $\widetilde{\mathbf{X}}$ be a partition of $\mathbf{X}$, and let $\le$ be a partial order on $\widetilde{\mathbf{X}}$. Then there exists the partial preference Q on $\mathbf{X}$ which generates the factor-set $\widetilde{\mathbf{X}}_Q=\widetilde{\mathbf{X}}$ with the same partial ordering of equivalence classes.

Consider one more tool for defining partial preference. Let $\mathbf{M}$ be a set of real functionals defined on $\mathbf{X}$. For each $f\in\mathbf{M}$ build symmetric and asymmetric parts of Q_f as defined in (1). Then construct sets

$$Q_{\mathbf{M}}^s=\bigcap_{f\in\mathbf{M}} Q_f^s,\quad Q_{\mathbf{M}}^a=\bigcap_{f\in\mathbf{M}} Q_f^a,\quad Q_{\mathbf{M}}=Q_{\mathbf{M}}^s+Q_{\mathbf{M}}^a. \qquad (2)$$

Lemma 4. Let $\mathbf{M}$ be a set of functionals defined on $\mathbf{X}$. Then relation $Q_{\mathbf{M}}$ defined by (1), (2) is a partial preference on $\mathbf{X}$.

Conditional conversion of this lemma constitutes a representation theorem for partial preference, see section 4.

3. Regular preference

There is a partial order on the set of distribution functions $\mathbf{P}$ which is called *stochastic dominance*. For $F, G \in \mathbf{P}$ we would say that G stochastically dominates F and denote this by $F \le G$ if $F(x) \ge G(x)$ for all $x \in \mathbf{R}$. Equivalent definition in terms of inverse distribution functions would sound $F^{-1}(v) \le G^{-1}(v)$ for all $v \in [0,1]$. Note that $W_a < W_b$ for $a < b$ and $W_{l(F)} < F < W_{u(F)}$ for any non-degenerate distribution function $F \in \mathbf{P}$.

For a non-decreasing sequence of distribution functions $\mathbf{F} = \{F_n \in \mathbf{P}, n = 1,2,\ldots\}$ with uniformly bounded supports there exists a limit $\lim_{n\to\infty} F_n = \sup \mathbf{F} = \overline{\mathbf{F}}$. The limit may be calculated by $\overline{\mathbf{F}}(x) = \lim_{n\to\infty} F_n(x) = \inf_n F_n(x)$. If the sequence in non-increasing then the limit is $\lim_{n\to\infty} F_n = \inf \mathbf{F} = \underline{\mathbf{F}}$ and may be calculated by $\underline{\mathbf{F}}(x) = \lim_{n\to\infty} F_n(x) = \sup_n F_n(x)$.

Now introduce a concept of regular preference relation.

Definition 1. A (partial) preference relation $\prec=$ on $\mathbf{P}$ is called *lower semicontinuous* if for any non-increasing sequence $F_n \in \mathbf{P}, n = 1,2,\ldots$ with $G \prec= F_n, n = 1,2,\ldots$ the following is true: $G \prec= \lim_{n\to\infty} F_n$. The preference is called *upper semicontinuous* if for any non-decreasing sequence $F_n \in \mathbf{P}, n = 1,2,\ldots$ with $F_n \prec= G, n = 1,2,\ldots$ the following is true: $\lim_{n\to\infty} F_n \prec= G$.

Definition 2. A (partial) preference relation $\prec=$ on $\mathbf{P}$ is called *regular* if it is a continuation of stochastic dominance and is lower and upper semicontinuous.

Later on in this paper we will consider only regular preferences $\prec=$ on $\mathbf{P}$.

4. Representation of regular preferences

Definition 3. We say that a functional $f : \mathbf{P} \to \mathbf{R}$ *represents* a complete preference relation $\prec=$ on $\mathbf{P}$ if $F \prec= G \Leftrightarrow f(F) \le f(G)$, $F, G \in \mathbf{P}$. The representing functional f is called *canonical* if $f(W_a) = a$, $a \in \mathbf{R}$.

Lemma 5. Let a complete preference relation $\prec=$ on $\mathbf{P}$ be regular. Then each equivalence class $K \in \widetilde{\mathbf{P}}$ contains exactly one degenerate distribution function.

Theorem 1. Let a complete preference relation $\prec=$ on $\mathbf{P}$ be regular. Then there exists the canonical representing functional $f : \mathbf{P} \to \mathbf{R}$.

This theorem establishes a desired conversion of lemma 2 for regular preference on a set of risks. Thus if one knows a complete preference relation, it can be in principle quantified by a functional f. Note that such a representation is not unique; any strictly increasing transform of a representing functional is a representing functional itself. Choosing a single representation may be achieved via canonical scheme described in [3].

In practice one never knows a complete preference, only a partial one. Next we show how one can represent partial preferences as well, and use the representation for approximation of the desired complete preference.

Definition 4. Let $\mathbf{M}$ be a family of functionals defined on $\mathbf{P}$. We say that this family *represents* a partial preference relation Q on $\mathbf{P}$, if $Q^s = Q^s_{\mathbf{M}}$ and $Q^a = Q^a_{\mathbf{M}}$, where $Q^s_{\mathbf{M}}$ and $Q^a_{\mathbf{M}}$ were defined in (1), (2).

Note that (1), (2) may be treated as follows: a pair of distributions (F,G) appears in the symmetric part of Q if all functionals in $\mathbf{M}$ are concordant on this pair in the sense $f(F)=f(G)$, $f\in\mathbf{M}$; the pair appears in the asymmetric part of Q if all functionals in $\mathbf{M}$ are concordant on this pair in the sense $f(F)<f(G)$, $f\in\mathbf{M}$. In all other cases distributions F and G are incomparable.

Theorem 2. Let a regular partial preference Q on $\mathbf{P}$ be a restriction of a regular complete preference R on $\mathbf{P}$ with representing functional f_R. Then there exists a representing family of canonical functionals $\mathbf{M}$ such that $f_R\in\mathbf{M}$.

Remark 1. Note that theorem 2 implicitly contains a method for sequential approximation of complete preference by partial ones. Indeed, let $Q_1,Q_2,\ldots$ be a sequence of partial preference relations on $\mathbf{P}$ such that each subsequent relation is a continuation of the previous one, id est, $Q_k^s\subseteq Q_{k+1}^s\subseteq\cdots\subseteq R^s$ and $Q_k^a\subseteq Q_{k+1}^a\subseteq\cdots\subseteq R^a$ for $k=1,2,\ldots$. Let $\mathbf{M}_k$ be a representing family of canonical functionals for preference Q_k, $k=1,2,\ldots$. Clearly $\mathbf{M}_k\supseteq\mathbf{M}_{k+1}\supseteq\cdots\supseteq\{f_R\}$, moreover, it can be shown that $\bigcap_{k=1}^{\infty}\mathbf{M}_k=\{f_R\}$. So building a sequence of continuations Q_k, $k=1,2,\ldots$ one approximates the functional of interest f_R by a sequence of functional families $\mathbf{M}_k$, $k=1,2,\ldots$.

Remark 2. Representing family of canonical functionals mentioned in theorem 2 is not uniquely defined, as shown in example 3 below. Theorem 2 provides constructing the *maximal* (with respect to inclusion) family of functionals, which is the desired property for approximation procedure described above.

5. Examples

Consider a few examples illustrating theory presented herein.

Example 1. Consider a functional $f(F)=\mathbf{E}F$, $F\in\mathbf{P}$. Clearly the functional is canonical. It generates the preference relation on $\mathbf{P}$ such that $F\sim G$ iff $\mathbf{E}F=\mathbf{E}G$ and $F\prec G$ iff $\mathbf{E}F<\mathbf{E}G$. This preference relation corresponds to a *risk-neutral* decision-maker. She makes decisions based on means of distributions, disregarding risk. This preference relation is regular. Its symmetric part generates the partition of $\mathbf{P}$ to equivalence classes

$$K_\lambda=\{F\in\mathbf{P}:\mathbf{E}F=\lambda\},\ \lambda\in\mathbf{R},\quad\text{so that}\quad\mathbf{P}=\sum_{\lambda\in\mathbf{R}}K_\lambda\,. \tag{3}$$

Each class K_λ contains exactly one degenerate distribution W_λ. Linear order on the factor-set is given by $K_\lambda<K_\mu$ iff $\lambda<\mu$.

Example 2. Let us construct discontinuous preference relations on $\mathbf{P}$. We start with the factor-set of example 1. Now break K_0 to two non-empty disjoint classes K_{01} and K_{02} so that $W_0\in K_{01}$. Define a linear order on this new partition $\widetilde{\mathbf{P}}=\{K_\lambda,\lambda\neq 0\}+K_{01}+K_{02}$ of $\mathbf{P}$ by

$$K_\lambda<K_{01}<K_{02}<K_\mu\ \text{ for }\lambda<\mu \tag{4}$$

and preserving the order for original classes. By lemma 1 the partition together with the linear order generates the preference relation Q on $\mathbf{P}$. Consider decreasing sequence of distribution functions $F_n = W_{1/n}, n = 1,2,\ldots$. For any $G \in K_{02}$ we have $G \prec F_n, n = 1,2,\ldots$, but $G \succ W_0 = \lim_{n\to\infty} F_n$. This means lack of lower semicontinuity of the preference relation Q.

Constructing a preference without upper semicontinuity is similar with ordering

$$K_\lambda < K_{02} < K_{01} < K_\mu \text{ for } \lambda < \mu$$

instead of (4) and the sequence $F_n = W_{-1/n}, n = 1,2,\ldots$.

Example 3. Consider stochastic dominance as a partial preference on $\mathbf{P}$. Let functionals $f_U : \mathbf{P} \to \mathbf{R}$ be defined by $f_U(F) = U^{-1}\left(\int_{l(F)}^{u(F)} U(x)dF(x)\right)$, where $U : \mathbf{R} \to \mathbf{R}$ is an increasing function. f_U is clearly canonical. Denote $\mathbf{U}$ the family of all such functionals. It may be shown that $\mathbf{U}$ represents stochastic dominance, however this family is not maximal. For example, it does not include coherent risk measures [4], in particular distorted probability functionals [5]. It also does not contain convex risk measures [6]. In fact this family does not contain functionals representing nonlinear preference relations. The maximal family for this relation, described in theorem 2, is so wide, that it is next to impossible to even imagine it. This relates to the fact that stochastic dominance is the minimal regular preference relation on $\mathbf{P}$.

Conclusion

Representation of complete and partial preferences by functional families has been considered in the paper. Representation theorems have been established, which in particular provide approximation methods for true but unknown complete preference. The results presented herein bring a tool for constructing efficient decision-making algorithms with estimation of errors due to uncertainty of underlying preference.

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