NONLINEAR PORTFOLIO ANALYSIS AND RESOURCES ALLOCATION

A.A. Novosyolov
Institute of computational modeling SB RAS
Krasnoyarsk state university
anov@icm.krasn.ru

Abstract

The paper is devoted to allocation of restricted resources under probabilistic uncertainty (risk). An example showing inability of classic second order theory to handle the problem is presented. Contemporary portfolio methods are being demonstrated to overcome the difficulties.

Introduction

A problem of portfolio choice (or allocation of restricted resources) may be informally represented as follows. Given a number of investment opportunities (or investment tools), each of which would bring uncertain return; how should one allocate resources (capital) to attain the best in some sense result. A similar problem arises in decision-making, where one needs to choose among a number of projects, each bringing uncertain income. Since uncertainty is usually modeled by probabilistic methods, the problem reduces to comparison of probability distributions.

In classic portfolio analysis distributions are compared using only the first two moments: mean and variance, or, in multi-dimensional case, mean vector and covariance matrix. This approach may be called the second order theory (by the maximal order of moments used); it stemmed from the papers of Markowitz and Tobin, and was logically completed for static framework in the Sharpe's CAPM model. Black, Scholes and Merton used dynamic variant of the approach for option price calculation. The second order theory may be also called the linear theory by the structure of methods used in it.

The linear theory works fine if the joint distribution of tools is normal; the normality assumption explicitly or implicitly appears in all the papers mentioned above. However, empirical research shows that distributions of real stock market instruments, credit portfolios of banks and other investment tools are usually far from normal. In particular, distribution of return of credit portfolios exhibits strong asymmetry and heavy tails [1], and joint distributions of assets returns involve nonlinear dependences [2]; the effects are inconsistent with normality assumption.

The expected utility method [3] is an efficient tool for overcoming the difficulties. The method allows processing asymmetric distributions, and catching some nonlinear dependence among instruments. However, the expected utility functional is itself linear with respect to mixture of distributions; this prevents from using the method in case individual preferences are nonlinear; experiments [4] show that this is often the case.

Further improvement of methods for investment decision-making may be achieved by using recent nonlinear functionals [5], and by using individual preference relations as input information for building decision-making methods [6].

The present paper uses a simple example to demonstrate disadvantages of the second order theory, and illustrates using expected utility and distorted probability functionals for building investment portfolio.

An example of tools distribution

Consider the random variables X_1 and X_2 with the following Bernoulli distributions:

$$P\{X_1 = 0\} = 0.1,\ P\{X_1 = 10\} = 0.9,\ P\{X_2 = 8\} = 0.9,\ P\{X_2 = 18\} = 0.1. \quad (1)$$

Probability functions (discrete “density functions") of the variables are depicted in the following figure.

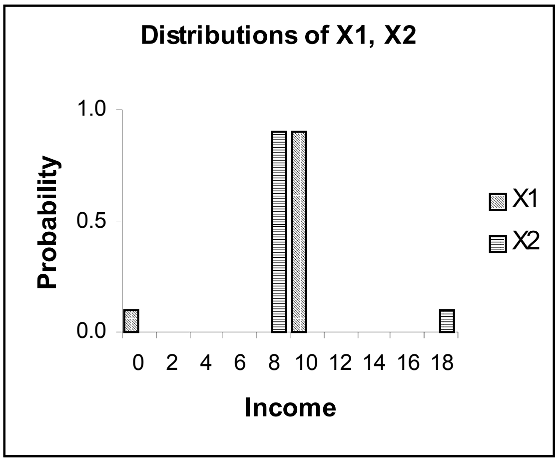

Fig. 1 Distributions of tools X_1 and X_2

The example allows the following treatment. Suppose that the individual possesses the wealth of 10 units, which may be completely destroyed as a result of an adverse event, which may occur during the upcoming year with probability 0.1. If the individual does nothing, then her wealth a year from now is described by the random variable X_1. The individual may insure her wealth by paying the insurance premium (1 unit), which reduces her wealth to 9 units; the latter would not change as a result of adverse event, since insurance company will cover the loss if it occurs. Suppose now that insurance is sold together with a lottery, which costs 1 unit and may bring the gain of 10 units with probability 0.1. If the individual would buy the insurance and the lottery, her wealth one year from now will be described by the random variable X_2.

Thus the property owner faces a problem: which of the variants X_1, X_2 is better. Most people would choose the project that brings X_2 (note that some would avoid activities, and choose X_1). A simple calculations show that expected values and variances of both random variables are equal:

$$\mathbf{E}X_1 = \mathbf{E}X_2 = 9,\ \mathbf{D}X_1 = \mathbf{D}X_2 = 9, \quad (2)$$

so second order theories would not help in selecting between the projects; they just would treat them as equivalent, in spite of the fact that most rational investors make clear distinction. Risk averse investors would rather choose the project X_2, since it does not bring large losses at all. In what follows we will consider how second order and other methods make choice between the projects (essentially, between the probability distributions), and study some methods of building optimal portfolios of the tools, represented by the random vector (X_1, X_2):

$$X_y = yX_1 + (1-y)X_2, \quad y \in \mathbf{Y}. \tag{3}$$

Here y denotes the weight of the first tool in the portfolio, and admissible set $\mathbf{Y} \subseteq \mathbf{R} = (-\infty, \infty)$ may be of the form $[0,1]$ or $\mathbf{R}$, depending on the essence of the problem. In particular, when one is to choose between the projects, the admissible set consists of just two points: $\mathbf{Y} = \{0,1\}$.

Note that the distribution of the portfolio (3) return depends not only on marginal distributions of (1), but also on the structure of dependence between the components of the random vector (X_1, X_2), which in our case may be described by the single parameter $m = \mathbf{P}\{X_1 = 10, X_2 = 18\}$. Values of the parameter cannot be arbitrary probabilities, since they should reside between the Frechet bounds, which in this case are $0 \le m \le 0.1$. The value $m = 0.09$ brings distribution with independent components, while boundary values of m correspond to comonotonic ($m = 0.1$) and anticomonotonic ($m = 0$) dependence; in the last case we have $X_1 + X_2 = 18$ with probability 1, and the correlation of (X_1, X_2) equals minus 1. Note that maximal correlation corresponding to the value $m = 0.1$ is significantly less than 1.

Second order method

Since expectations of X_1 and X_2 coincide, the second order method reduces to minimization of the variance of X_y, which equals to

$$s(y) = DX_y = 4(y^2 - y)(9 - 50m) + 9.$$

Minimization of the function s in y gives $y^* = 1/2$, an anticipated value in the symmetric problem. Minimal variance is equal to $s(y^*) = 50m$, and, in particular, equals to 0 for anticomonotonic tools. Note that diversification is attained even in the case of maximal positive correlation of tools, $m = 0.1$; the variance of the optimal portfolio is equal to 5, which is almost two times less than the variance of each tool. The next figure presents dependence of portfolio X_y variance on the weight y under different values of m.

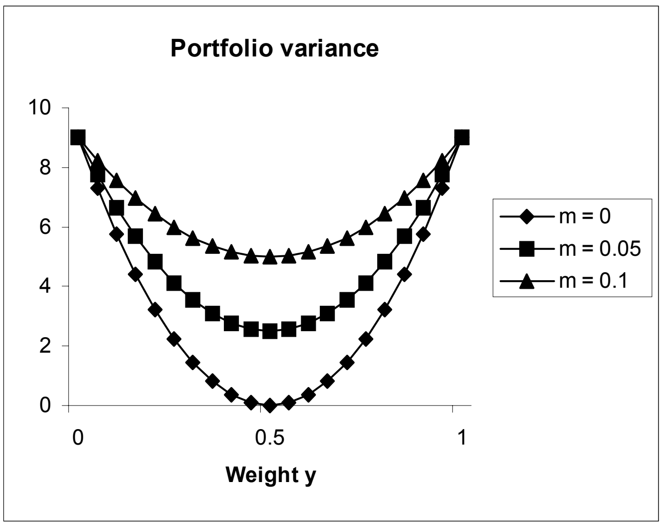

Fig. 2. Criterion in the second order method

Note the symmetry of the criterion form with respect to the tools X_1, X_2. In the following nonlinear methods such symmetry is rather exclusion than rule.

Expected utility maximization

Now let us apply the expected utility method. Let U be a real utility function. If U is increasing and strictly concave, that is,

$$U(\lambda x + (1-\lambda)y) > \lambda U(x) + (1-\lambda)U(y), \quad x, y \in R, \quad \lambda \in (0,1),$$

then U describes a risk averse investor [7]. Elements of the exponential class

$$U(x) = (1 - e^{-\alpha x})/(1 - e^{-\alpha}), \qquad x \in R \tag{4}$$

with $\alpha > 0$ are examples of such utility functions. Given a utility function U, an expected utility $\rho(X)$ of a project (portfolio) X is calculated as

$$\rho(X) = EU(X) = \int_{-\infty}^{\infty} U(x) dF_X(x),$$

where F_X is a distribution function of the random variable X. In particular, for discrete random variables taking values x_k with probabilities $p_k = P\{X = x_k\}$, $k = 1,...,N$, one has

$$\rho(X) = \sum_{k=1}^{N} U(x_k) p_k . \tag{5}$$

Thus the problem of project choice reduces to comparison of expected utilities $\rho(X_1)$, $\rho(X_2)$ of the projects X_1, X_2. It turns out that an investor whose preferences are described by a utility function (4) with some parameter $\alpha > 0$ would prefer the project X_2. The next figure illustrates dependence of the difference $\rho(X_2) - \rho(X_1)$ on the value of the parameter α.

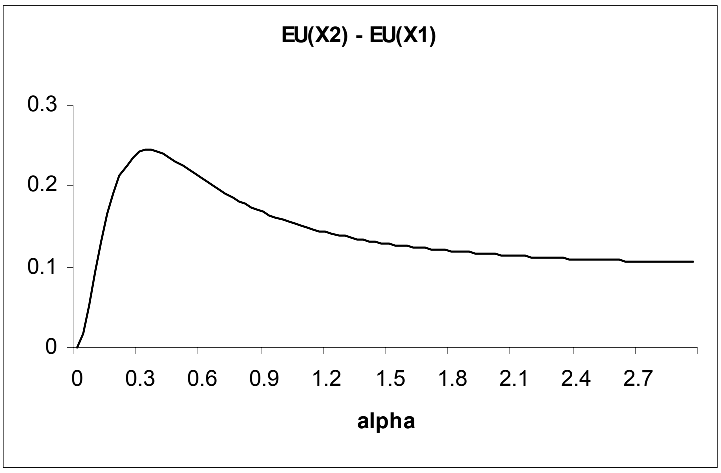

Fig. 3. Dependence of the difference of expected utilities on α

Next let us use the expected utility method with exponential utility functions (4) for portfolio (3) selection. We need to find a weight value y, which provides $\max_y \mathbf{E}U(X_y)$. Explicit representation for the expected utility can be easily obtained from (5). We will study dependence of a solution on parameters α, m.

Note, that since utility function U is monotone, maximizing the expected utility of a portfolio is the same as maximizing its certain equivalent, which for any random variable X has the form

$$c(X) = U^{-1}(\mathbf{E}U(X)),$$

where U^{-1} is an inverse function for U. Thus quantities ρ and c are interchangeable, and using one or another is a matter of convenience.

Next, note that the limiting case $\alpha = 0$ corresponds to the linear utility function $U(x) \equiv x$, which describes a risk-neutral investor [7]. Such an investor maximizes the mean value of portfolio return $\mathbf{E}X_y$, which in our case is constant and equal to 9.

Another limiting case corresponds to the value $m = 0$. In this case the instruments X_1, X_2 are ideally negatively correlated (i.e. they are linearly dependent: $X_2 = 18 - X_1$), so mixing them with weight $y = 1/2$ provides the ideally diversified portfolio with degenerate distribution of return $X_{1/2} = 9$. Such a portfolio is optimal for any investor with utility function from the class (4).

Given $\alpha > 0$, a utility function (4) describes a risk averse investor. If $m > 0$, then ideal correlation of instruments is impossible, and maximizing expected utility always leads to a nontrivial solution, which depends both on risk aversion α, and dependence of the instruments m.

The next figure presents graphs of certain equivalent for a number of values of α.

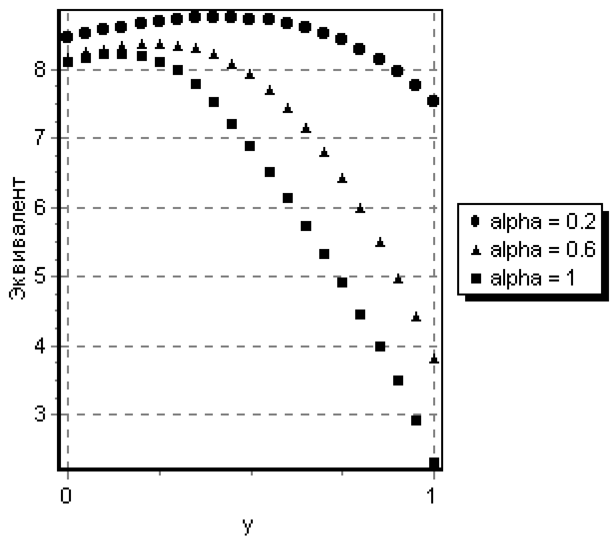

Fig. 4. Certain equivalent of a portfolio

When risk aversion is small (α = 0.2), the values of the goal function exhibit insignificant dependence of the weight y, and the graph is almost symmetric. When risk aversion increases, the graph becomes more asymmetric, and more capital is to be invested into the second instrument X_2 in the optimal portfolio.

Figure 5 depicts graphs of optimal portfolio weight y vs. risk aversion α for a number of values of the parameter m.

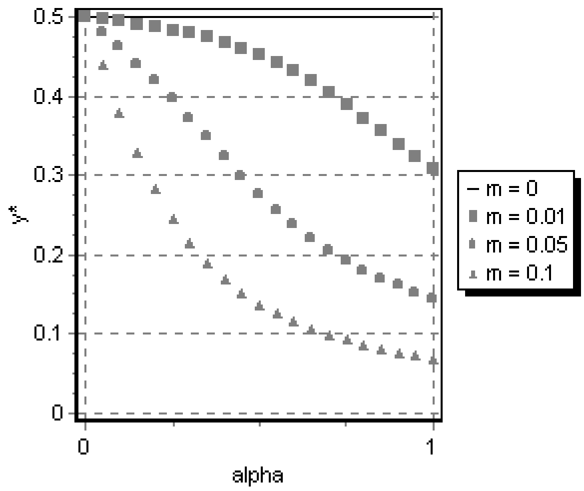

Fig. 5. Dependence of capital allocation on risk aversion.

It is worth noting that the weight y is always less than 1/2, that is, the second instrument X_2 is always included into the optimal portfolio with greater weight; as if it is more preferred than X_1.

Distorted probability method

Consider the distorted probability method, which is maximizing a distorted probability functional on a given set of distributions. The functional is calculated as [5], [6]:

$$\pi(F) = \int_{-\infty}^{0} [1 - g(1 - F(x))]dx + \int_{0}^{\infty} g(1 - F(x))dx = \int_{0}^{1} F^{-1}(1 - v)dg(v),$$

where F is a distribution function, g is a parameter, a monotone distortion function, defined on [0,1], and possessing properties $g(0) = 0$, $g(1) = 1$, $g(x) \le x$, $x \in [0,1]$. In particular, for a discrete distribution function F, having jumps at the points x_k, (written in ascending order) of size p_k, $k = 1,\dots,N$, the following representations hold [6]:

$$\pi(F) = \sum_{k=1}^{n} x_k \left[g\left(\sum_{i=k}^{n} p_i \right) - g\left(\sum_{i=k+1}^{n} p_i \right) \right] = \sum_{k=1}^{n} (x_k - x_{k-1}) g\left(\sum_{i=k}^{n} p_i \right), \qquad (6)$$

where empty sum is set to 0, and $x_0 = 0$. Note that the value of the functional on the distribution of the portfolio (3) will be a piecewise-linear function of y.

Consider a class of power distortion functions

$$g(v) = v^{\alpha},\ \alpha > 1.$$

Here the parameter α indicates risk aversion. If $\alpha = 1$, the functional coincides with the expected value of a distribution, so the functional describes risk-neutral investor. Risk averse investor would have $\alpha > 1$.

Consider the problem of selecting a project. Calculating the values of the functional π on distributions F_1, F_2 of the projects X_1, X_2 from our example is easily implemented using (6). We will present here the graph of the difference $\pi(F_2) - \pi(F_1)$:

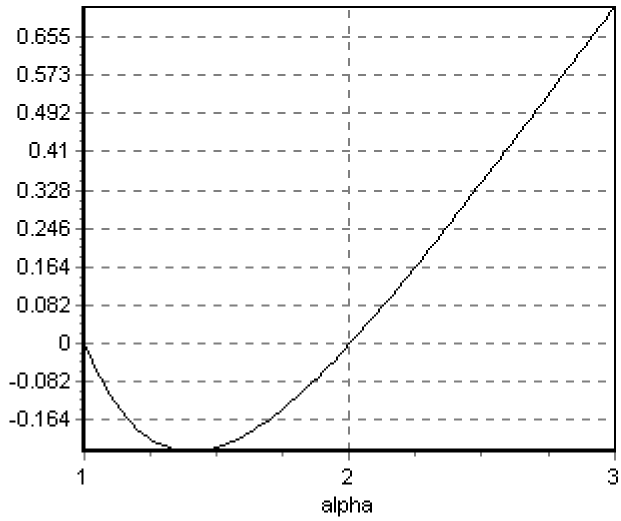

Fig. 6. Difference of values of the distorted probability functional

One can see that risk averse investors behave differently. Investors with modest risk aversion ($1 < \alpha < 2$) would choose the project X_1, while more risk averse investors ($\alpha > 2$) would prefer X_2. This strongly nonlinear behavior of distorted probability functional allows catching nonlinear properties of real preferences, which were pointed out in [4].

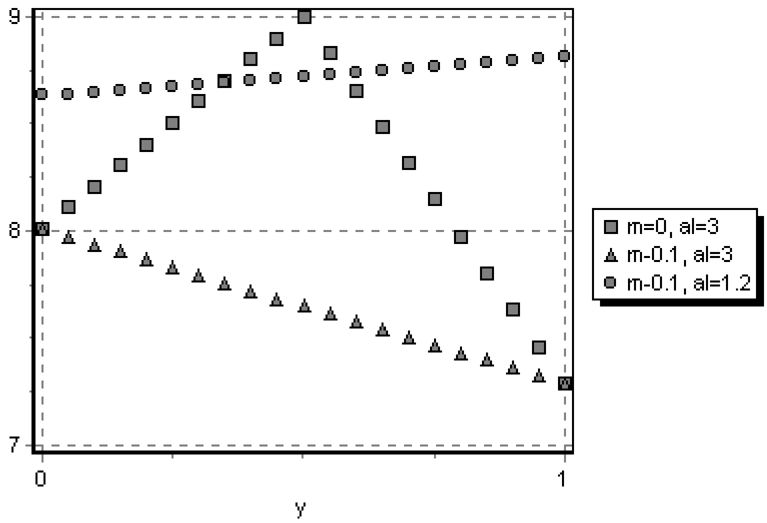

Fig. 7. Distorted probability of a portfolio

Now turn to portfolio X_y selection, defined in (3). This problem reduces to maximizing the distorted probability functional as a function of weight y. Figure 7 depicts a few graphs of this optimization criterion for a number of values of parameters α and m. One can see that optimal portfolio consists either of a single instrument, or is an equiweighted mixture. The fact can be formally proven for all parameter values $\alpha > 1$, $0 \le m \le 0.1$. In the limiting case of risk-neutral investor ($\alpha = 1$), the functional value is constant and is equal to expected value of a portfolio. In the limiting case of ideal negative correlation between the instruments ($m = 0$), the optimal portfolio includes instruments with equal weights ($y = 0.5$).

Conclusion

The present paper uses a simple example to demonstrate inability of the classic second order theories to represent human preferences in the presence of asymmetric distributions. The two contemporary methods, expected utility and distorted probability, have been presented and studied using the same example.

Pointing to disadvantages of the second order theories does not mean negation of their role for both theory and applications. On the contrary, Markowitz and Sharpe's models will always form a solid theoretical base for deriving more complex methods. As for applications, using high-order methods may be undesirable when there is not enough information. So the theory would benefit from developing methods of different classes, including those of the second order. In applications it is a good idea to select the most appropriate method in each case, depending on the amount of information available.

The author appreciates intensive discussions of risk theory topics with the FAM Seminar participants, which dramatically influence the author's views, and, in particular, allow improving this paper.

A Windows program that allows reproducing the experiments described in the paper is available for downloading at

http://www.geocities.com/novosyolov/download.htm#NonlinIll

References

1. *Credit Metrics - technical document*. Program & Co., New York, 1997, 200p.
2. McNeil A.J., Frey R. Estimation of tail-related risk measures for heteroskedastic financial time series: an extreme value approach. *J. of Empirical Finance*, **7** (2000), 271-300.
3. Neumann, J. Von, Morgenstern O. *Theory of Games and Economic Behavior.*. Princeton University Press: Princeton, 1953.
4. Cox L.A. *Risk Analysis: Foundations, Models and Methods*. Kluwer, 2001, 568 p.
5. Wang S. Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin*, **26** (1996), 1, 71-92.
6. Novosyolov A.A. *Mathematical Modeling of Financial Risks: Measuring Theory.* Novosibirsk: Nauka, 2001, (in Russian)
7. Pratt J.W. Risk Aversion in the Small and in the Large. *Econometrica*, **32** (1964), p. 122-136.