GENERALIZED COHERENT RISK MEASURES IN DECISION-MAKING UNDER RISK

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Coherent risk measures have lately become a hot topic in both theoretical research and practical applications. Since they possess a number of disadvantages, some generalizations and modifications have been proposed recently. In the present paper we propose one more generalization which holds a number of attractive properties. A representation theorem for the risk measures class has been obtained, properties of functionals have been established, partial cases and examples are also presented.

Key words and phrases: risk, coherent risk measure, conditional value at risk, admissible risks set, representation theorem

1. Introduction

Coherent risk measures were introduced in [1] and had immediately become a very popular tool for decision-making under risk. Simultaneously their disadvantages were also examined, thus provoking modifications and generalizations in a number of directions. Convex risk measures [2] perhaps were the first generalization of the sort. They are constructed by replacing subadditivity and positive homogeneity properties by a more general convexity property.

Another generalization attempt was made in [3]. The authors pointed to the fact that moving a risk into admissible set may be implemented not only by adding a certain position, as in definition of coherent risk measures, but also buy adding appropriate risky position. A norm value of the smallest risky position of the sort becomes the value of this new risk measure. As it was pointed out in [3], this way one can obtain less expensive tools for risk control.

Unfortunately the functionals of the class introduced in [3] do not represent any reasonable preference relation on a set of risks. Besides that, they do not generalize coherent risk measures in the strict sense. This is due to the fact that the functionals of the new class are identically equal to zero on sets of admissible risks, that is, they do not distinguish between admissible risks. This is often sufficient for regulator bodies, whose goal is preventing downside risks. However this is inappropriate for making business decisions, where the goal is choosing the best risk among admissible ones. To solve such problems one needs risk measures which distinguish between admissible risks. Thus appropriate generalization of coherent risk measures is very important.

In [4] one such generalization was considered using Euclidean norm in $\mathbf{R}^n$. The present paper is devoted to the case of general norm in $\mathbf{R}^n$. The axiomatic description in terms of sets of admissible risks is presented. The representation theorem for the functionals is obtained, which allows efficient calculation of risk measures. Properties of proposed risk measures are also studied.

Introduce notation. Let $(\Omega, \mathbf{A}, \mathbf{P})$ be a probability space. We will call $\mathbf{P}$ "physical" measure and would not fix it in advance. To simplify the following we will assume that the event space is finite: $|\Omega| = n$. In this case one can choose the largest σ-algebra $\mathbf{A} = 2^{\Omega}$, and represent probability measures Q on the measurable space $(\Omega, \mathbf{A})$ by elements of $\mathbf{R}^n$, or, to be precise, by elements of its standard simplex

$$S_n = \{Q = (q_1, \dots q_n) \in \mathbf{R}^n : q_1 \geq 0, \dots q_n \geq 0, q_1 + \cdots + q_n = 1\} \quad (1)$$

A random variable (or risk) X on $(\Omega, \mathbf{A})$ is any mapping from Ω to $\mathbf{R}$, which is always measurable in this case. Denote $\mathbf{X}$ the set of all risks on $(\Omega, \mathbf{A})$; it is clear that $\mathbf{X}$ is isomorphic to

$\mathbf{R}^n$. Numbering the elements of Ω in some arbitrary order: $\Omega = \{\omega_1, \ldots \omega_n\}$, denote $X(\omega_i) = x_i$, $i = 1, \ldots, n$, and identify risks $X \in \mathbf{X}$ with vectors $X = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Introduce the order on $\mathbf{R}^n$ as usual: $X = (x_1, \ldots, x_n) \leq Y = (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for all $i = 1, \ldots, n$. We will also write $X < Y$ if $x_i < y_i$ for all $i = 1, \ldots, n$. Denote C_+ the cone of nonnegative random variables, and C_{--} - the cone of negative random variables, that is

$$C_+ = \{X \in \mathbf{X} : X \geq 0\},\ C_{-} = X \in \mathbf{X} : X \geq 0 .$$

Definition 1. A set $A \subseteq \mathbf{X}$ is called a set of admissible risks, if it is a closed convex cone in the linear space $\mathbf{X}$ and

$$A \cap C_{--} = \varnothing,,\ C_+ \subseteq A. \tag{2}$$

One can introduce norm in $\mathbf{R}^n$ in many ways, for example, a norm $\|\cdot\|_p$ for some $1 \leq p < \infty$ is defined by

$$\|X\|_{=} = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p},\ X = (x_1, \ldots, x_n) \in \mathbf{R}^n, \tag{3}$$

and

$$\|X\|_{\infty} = \lim_{p \to \infty} \|X\|_p = \max\{|x_1|, \ldots, |x_n|\},\ X = (x_1, \ldots, x_n) \in \mathbf{R}^n. \tag{4}$$

Recall [5] that preference relation $\prec$ on $\mathbf{X}$ is any complete transitive binary relation and risk measure is any functional $f : \mathbf{X} \to \mathbf{R}$. We say that risk measure f represents the preference relation $\prec$ if

$$f(X) \leq f(Y) \Leftrightarrow X \prec Y,\ X, Y \in \mathbf{X}. \tag{5}$$

The paper is organized as follows. In section 2 the classical coherent risk measure and its properties are presented. In section 3 we briefly consider a generalization of coherent risk measures from [3] and point to its disadvantages. In section 4 another generalization of coherent risk measures is proposed, its properties are studied and a representation theorem is obtained.

2. Coherent risk measures

Coherent risk measures were introduced in [1]. Following [6] we will equivalently describe the topic in terms of supermodular functions which are simply negatives of coherent risk measures; this is just more convenient for our purpose. To be precise, if $\rho : \mathbf{X} \to \mathbf{R}$ is coherent risk measure in the sense of [1], then we will call coherent risk measure a functional $f : \mathbf{X} \to \mathbf{R}$, which is related to ρ by $f(X) = -\rho(X)$, $X \in \mathbf{X}$.

Definition 2. Coherent risk measure is a functional $f : \mathbf{X} \to \mathbf{R}$ possessing the properties of monotonicity, superadditivity, positive homogeneity, and shift invariance, respectively:

$$X \leq Y \Rightarrow f(X) \leq f(Y), \tag{6}$$
$$f(X + Y) \geq f(X) + f(Y), \tag{7}$$
$$f(\lambda X) = \lambda f(X), \lambda > 0, \tag{8}$$

$$f(X + aI) = f(X) + a, a \in \mathbf{R}. \tag{9}$$

Here X, Y are arbitrary risks in $\mathbf{X}$ and $I = (1,\dots,1)$. Note that (8) implies

$$f(0) = 0, \tag{10}$$

which together with (6) gives

$$f(X) \geq 0 \text{ for } X \geq 0. \tag{11}$$

In [1] authors obtained a characterization of coherent risk measures in terms of sets of admissible risks. First note, that if a risk measure is defined, then the set of admissible risks is naturally defined by

$$A_f = \{X \in \mathbf{X} : f(X) \geq 0\}. \tag{12}$$

In [1] it was shown that for a coherent risk measure f the set of admissible risks coincide with the object defined in definition 1. Exactly, the following theorem is valid.

Theorem 1. Let f be a coherent risk measure on $\mathbf{X}$, that is, a real functional possessing the properties (6) - (9), and let A_f be defined in (12). Then A_f is a closed convex cone in $\mathbf{X}$ and $C_+ \subseteq A_f$, $A_f \cap C_{--} = \varnothing$.

Such coherent risk measures may be characterized by sets of admissible risks.

Theorem 2. Let A be a set of admissible risks in the sense of definition 1, and let the functional f_A be defined on $\mathbf{X}$ by

$$f_A(X) = \sup\{b \in \mathbf{R} : X - bI \in A\}. \tag{13}$$

Then f_A possesses the properties (6) – (9), that is, f_A is a coherent risk measure.

Example 1. A simple but important example of coherent risk measure is expectation with respect to a given probability measure on $(\Omega, \mathbf{A})$. Consider a probability measure $Q = (q_1,\dots,q_n)$ in S_n, where the standard simplex S_n was defined in (1). Then the functional f_Q

$$f_Q(X) = \mathbf{E}_Q X = q_1 x_1 + \cdots + q_n x_n \text{ for } X = (x_1,\dots,x_n) \in \mathbf{R}^n \tag{14}$$

is a coherent risk measure. Other examples are distorted probability functional [7] and its partial case CVaR [8].

In [1] a representation theorem for coherent risk measures was also obtained. It represents the functional as the lower bound of a set of linear functionals.

Theorem 3. Let f be a coherent risk measure on $\mathbf{X}$. Then there exists a family of probability measures $\mathbf{Q} = \mathbf{Q}_f$ in S_n such that

$$f(X) = \inf_{Q \in \mathbf{Q}} \mathbf{E}_Q X, \; X \in \mathbf{X}. \tag{15}$$

The family $\mathbf{Q}$ may be called a generator of a risk measure f. The coherent risk measure of example 1 (expectation) is represented in the form (15) with a single-point family $\mathbf{Q}$.

Remark 1. The family $\mathbf{Q}$ is defined not uniquely. Actually together with any family $\mathbf{Q}$ its convex hull defines in (15) the same functional.

3. Jarrow and Purnanandam generalization.

In [3] the following generalization of coherent risk measures was proposed. Let $\|\cdot\|$ be a norm on $\mathbf{X}$. The distance from a point X to a set $B \subseteq \mathbf{X}$ is defined as usual by

$$d(X,B) = \inf_{Y \in B} \|X - Y\|.$$

Definition 3. Let A be a set of admissible risks. The functional

$$j_A(X) = -\inf_{Y \in A} \|X - Y\| = -d(X,A), \ X \in \mathbf{X}$$

is called a JP-generalized coherent risk measure.

Note that JP-measure is identically equal to 0 on the set of admissible risks:

$$j_A(X) = 0, \ X \in A$$

which means that it does not distinguish between admissible risks. This restricts using such functionals in decision-making applications. They allow avoiding inadmissible decisions, but fail to choose between admissible ones. Besides that, (17) means that a coherent risk measure is not a partial case of (16), because it does not have the property (17). Moreover, classical coherent risk measures distinguish between admissible risks and allow decision-making in both sides of risk spectrum.

The goal of the present paper is modification of Jarrow and Purnanandam approach in such a way, that the resulting risk measure is an actual generalization of coherent risk measure without losing its desirable properties.

4. Generalized coherent risk measure.

Fix a norm $\|\cdot\|$ on $\mathbf{X}$. Let A be a set of admissible risks. We will call the following functional

$$f(X) = f_A(X) = \delta_A(X) \inf_{Y \in \partial A} \|X - Y\| = \delta_A(X) d(X, \partial A) \qquad (18)$$

the generalized coherent risk measure. Here

$$\delta_A(X) = \begin{cases} 1, & X \in A \\ -1 & X \notin A \end{cases} \qquad (19)$$

stands for indicator function of A and ∂A denotes the border of A. It can be easily seen that (18) coincides with (16) for $X \in A^c$, but those are different for admissible risks $X \in A$.

Denote $\mathbf{X}^*$ the space of linear continuous functionals on $\mathbf{X}$ (the dual space), $\|\cdot\|_*$ - norm in this space, and consider the cone A^*, which is dual to A. This dual cone consists of all linear continuous functionals, which takes nonnegative values on all elements of A:

$$A^* = \{g \in \mathbf{X}^* : g(X) \geq 0, X \in A\} \qquad (20)$$

Consider a subset of functionals with unit norm:

$$A_1^* = \{g \in A^* : \|g\| = 1\}.$$

Theorem 4. Let f be a generalized coherent risk measure, defined by a set of admissible risks A. Then the following representation is valid:

$$f(X) = \inf_{g \in A_1^*} g(X),\ X \in \mathbf{X}. \tag{21}$$

We would not prove theorems here, but would describe some geometric properties of sets of admissible risks; this would allow better understanding of the new class of functionals. For a linear functional $g \in \mathbf{X}^*$ denote $L_g^+ = \{X \in \mathbf{X} : g(X) \geq 0\}$ and $L_g^- = -L_g^+ = \{X \in \mathbf{X} : g(X) \leq 0\}$ its half-spaces, and $L_g^0 = L_g^+ \cap L_g^- = \{X \in \mathbf{X} : g(X) = 0\}$ its hyperplane. Then

$$A = \bigcap_{g \in A^*} L_g^+ = \bigcap_{g \in A_1^*} L_g^+,\ \overline{A^c} = \bigcup_{g \in A^*} L_g^- = \bigcup_{g \in A_1^*} L_g^-,$$

and $d(X, L_g^0) = |g(X)|$ as long as $\|g\|_* = 1$. Note also that $A^* \subseteq C_+^*$, where C_+^* is the dual cone to C_+.

The following theorem states some properties of generalized coherent risk measures.

Theorem 5. Let f be a generalized coherent risk measure. Then it possesses the properties of monotonicity, superadditivity and positive homogeneity (6) – (8).

In general the functional does not have the shift invariance property in the form of (9) (though it has the one in a partial case). It has another similar property instead. To describe it we need some more notation. For $X \notin A$ denote $\pi(X)$ a point of ∂A which is closest to X. Next, denote $u(X) = (\pi(X) - X)/\|\pi(X) - X\|$ the unit vector in the direction of $\pi(X) - X$. Consider a line $\pi(X) + \lambda u(X), \lambda \in \mathbf{R}$ passing the points $X, \pi(X)$. Denote $K(X)$ the subset of that line, containing all the points, for which $\pi(X)$ is the closest point of ∂A. We see that $K(X)$ is not empty, since it contains X. Denote $\Lambda(X)$ the set of all $\lambda \in \mathbf{R}$ for which $\pi(X) + \lambda u(X) \in K(X)$. Clearly $\Lambda(X)$ either coincides with $\mathbf{R}$ or is an interval bounded from above. Denote $\lambda_m(X) = \sup \Lambda(X)$. We have $\lambda_m(X) \geq 0$ and, maybe, $\lambda_m(X) = \infty$.

Theorem 6. Let $f = f_A$ be a generalized coherent risk measure defined by a set of admissible risks A. Then for any $X \notin A$ we have $f(\pi(X) + \lambda u(X)) = \lambda$, $\lambda \in (-\infty, \lambda_m)$.

The following theorem states that coherent risk measures are actually partial case of generalized coherent risk measures (18).

Theorem 7. Let the norm $\|\cdot\|_\infty$ be defined in $\mathbf{R}^n$ by (4). Then the generalized coherent risk measure (18) coincides with the classical risk measure from [1].

Consider examples of generalized coherent risk measures. First let $A = C_+$. By (2) this means that A is the smallest possible set of admissible risks. It represents a very risk averse decision-maker, actually, the one avoids taking any risk at all. In this case the representation theorem 5 gives

$$f_A(X) = \inf_{g \in A_1^*} \sum_{k=1}^n g_k x_k = \min_k x_k .$$

Indeed, this person makes her decisions based only on the worst case scenario.

The partial case of Euclidean norm $\|\cdot\|_2$ in $\mathbf{R}^n$ is considered in detail in [4].

5. Conclusion

The class of generalized coherent risk measures was introduced and studied in the present paper. Generalization was implemented in the manner, similar to that of the paper [3], and was led to the very end. The resulting class of risk measures indeed contains the class of all coherent risk measures in the sense of [1].

The new risk measures were introduces via sets of admissible risks. A representation theorem was obtained, where the functional of interest is shown to be a lower bound of a family of linear continuous functionals. Some other properties of the new functional were also studied.

Axiomatic introduction of this class via properties of functionals is also worth considering; this problem will be studied separately.

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