ФУНКЦИЯ КОПУЛЫ, КАК СТРУКТУРА ЗАВИСИМОСТИ
COPULA FUNCTION AS A DEPENDENCE STRUCTURE

Arcady Novosyolov

Institute of computational modeling SB RAS
ICM SB RAS, Academgorodok, Krasnoyarsk, Russia, 660036
anov@icm.krasn.ru

Correlation structure of a multidimensional probability distribution captures only a small fraction of information on components dependence. To overcome the lack of information copula functions were intensively used during recent decades. The present paper gives an overview of copula functions as descriptors of dependence structure, presents some general classes of copula functions, and provides algorithms for using copula functions in decision-making under risk. Special attention is paid to discrete marginals case, which has not been addressed much in the literature before.

Keywords: probability distribution, Sklar's theorem, copula, dependence, Pearson and rank correlations, comonotonicity, Frechet bounds, Archimedean copula, discrete marginals, canonical copula.

Correlation has been used as a measure of dependence for over a century. Though copulas were first introduced in 1959 [1], they were not used in decision-making until the last decade, when multiple disadvantages and pitfalls of correlation were discovered, see eg. [2]. This led to a comprehensive research of copulas [3-5]. In the present paper we provide a brief survey of available results on copulas and their using in decision-making under risk. Special attention is being paid to the case of discrete marginals, which is not covered to full extent as yet. Detailed discussion of topics outlined here will be delivered at a conference session.

Consider the distribution function

$$F_{X_1,X_2,\ldots X_n}(x_1,x_2,\ldots x_n)=P(X_1\le x_1,X_2\le x_2,\ldots X_n\le x_n) \quad (1)$$

of n-dimensional random vector $X=(X_1,X_2,\ldots X_n)$ with marginal distribution functions

$$F_{X_k}(x_k)=F_{X_1,X_2,\ldots X_n}(\infty,\infty,\ldots,x_k,\infty,\ldots,\infty). \quad (2)$$

A function $C(u_1,u_2,\ldots,u_n)$ defined on $[0,1]^n$ is called a *copula function* if it is a distribution function of some n-dimensional random vector with uniform $[0,1]$ marginals.

The following theorem belongs to A. Sklar [1].

Sklar's Theorem. For any distribution function $F_{X_1,X_2,\ldots X_n}$ there exists a copula function C such that $F_{X_1,X_2,\ldots X_n}(x_1,x_2,\ldots x_n)=C(F_{X_1}(x_1),F_{X_2}(x_2),\ldots F_{X_n}(x_n))$ for all $x=(x_1,x_2,\ldots,x_n)\in R^n$. If the marginal distribution functions (2) are continuous then the copula function is unique.

This theorem suggests separating of description roles. Marginal distribution functions describe the standalone behavior of components $X_1,X_2,\ldots,X_n$, while copula function describes there joint behavior, thus representing dependence structure in its clearest form.

There are some special copula functions. The copula function

$$C^+(u_1,u_2,\ldots u_n)=\min\{u_1,u_2,\ldots u_n\} \quad (3)$$

is called the *upper Frechet bound*, and

$$C^0(u_1,u_2,\ldots u_n)=u_1u_2\cdots u_n \quad (4)$$

is called the *independent* copula. The *lower Frechet bound*

$$C^{-}(u_1,u_2,\ldots u_n)=\max\{u_1+u_2+\cdots+u_n-n+1,0\} \quad (5)$$

is a copula function only for $n=2$, yet it is important because of inequalities

$$C^{-}(u_1,u_2,\ldots u_n)\le C(u_1,u_2,\ldots u_n)\le C^{+}(u_1,u_2,\ldots u_n) \quad (6)$$

which hold for any copula function C and any n.

Consider special classes of copula functions.

Normal copulas. Let Σ be a symmetric positive definite matrix with unit diagonal elements (a correlation matrix). Denote Φ the (univariate) standard normal distribution function and Φ_Σ the multivariate normal distribution function with zero means, unit variances and correlation structure given by Σ. Then

$$C_N(u_1,u_2,\ldots u_n)=\Phi_\Sigma(\Phi^{-1}(u_1),\Phi^{-1}(u_2),\ldots,\Phi^{-1}(u_n)) \quad (7)$$

is called a normal copula with correlation structure Σ. Note that this copula produces multivariate normal distributions with normal marginals, but quite different distribution with non-normal marginals. Closely related to this class are so called *elliptic* copulas, including *Frank* and *Student* ones.

Archimedean copulas. A general tool for constructing copulas is building them from a *generator function*. A function $f:(0,1]\to(0,\infty)$ is called a generator function if it is twice differentiable, $f(0)=\infty$, $f(1)=0$, $f'(t)<0$, $f''(t)>0$, $t\in(0,1)$. The building algorithm for constructing an Archimedean copula C_f is presented by

$$C_f(u_1,u_2,\ldots u_n)=f^{-1}(f(u_1)+f(u_2)+\cdots f(u_n)),\ u=(u_1,u_2,\ldots u_n)\in[0,1]^n. \quad (8)$$

Next table shows some classical Archimedean copulas for $n=2$.

Table 1. Some classical Archimedean copulas.

Name	**Generator** $f(t)$	**Copula** $C(u_1,u_2)$
Independent	$-\ln t$	u_1u_2
Gumbel	$(-\ln t)^\alpha$	$\exp\left(-\left(u_1^\alpha+u_2^\alpha\right)^{1/\alpha}\right)$
Kimeldorf-Sampson	$t^{-\alpha}-1$	$\left(u_1^{-\alpha}+u_2^{-\alpha}-1\right)^{-1/\alpha}$

As Sklar's theorem states, copula function is not unique in case of discrete marginal distributions. To build a canonical copula we propose using piecewise uniform multivariate distributions.

References

1. Sklar A. Fonctions de repartition a n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Universite de Paris*, **8**, 229-231.
2. Embrechts P., McNeil A.J., Straumann D. Correlation and Dependency in Risk Management: Properties and Pitfalls. Department of Mathematik, ETHZ, Zurich, Working paper.
3. Nelsen R. *An Introduction to Copulas*. N.-Y., Springer, 1999.
4. Bouye E., Durrleman V., Nikeghbali A., Riboulet G., Roncalli T. Copulas for Finance: a Reading Guide and Some Applications. City Univ., London, 2000.
5. Wang S. Aggregation of Correlated Risk Portfolios: Models and Algorithms. CAS Committee on Theory of Risk. Report.